Existence of Solution to Generalized Multivalued Vector Variational-like Inequalities

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Abstract—In this paper, we consider a generalized multivalued vector variational-like inequality and obtain some existence results. Some special cases are also discussed.

Keywords—Generalized multivalued vector variational-like inequality; Existence result; Affine mapping; Lower semicontinuity

I. INTRODUCTION AND PRELIMINARIES

Let $Y$ be a Banach space with a convex cone $P$ such that $\text{int } P \neq \emptyset$ and $P \neq Y$, where int denotes the interior. We use the following vector ordering: for any $x_1, y_1 \in Y$, $y_1 < x_1$ if and only if $x_1 - y_1 \in -\text{int } P$ and only if $y_1 - x_1 \notin -\text{int } P$. Let $X$ be a nonempty subset of a Banach space $E$ and $Y$ a Banach space with a convex cone $P$ such that $\text{int } P \neq \emptyset$ and $P \neq Y$. Let $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ be a single-valued mapping, $M, S, T : X \to 2^{(E,Y)}$ be three set-valued mappings, where $L(E, Y)$ is the space of all linear continuous mappings from $E$ into $Y$, and $\{t, x\}$ denotes the value of $t$ at $x$. $\eta : X \times X \to Y : H : X \times X \to Y$ be two bimappings. We consider the following two generalized multivalued vector variational-like inequality, for short, denoted by GMVVLI-1 and GMVVLI-2, respectively.

GMVVLI-1: Find $y \in X$, such that for any

$$\langle N(u, v, w), \eta(x, y) \rangle + H(x, y) \notin 0, \ \forall x \in X.$$  

GMVVLI-2: Find $y \in X$, such that

$$\exists u \in M(y), v \in S(y), w \in T(y)$$

satisfying

$$\langle N(u, v, w), \eta(x, y) \rangle + H(x, y) \notin 0, \ \forall x \in X.$$  

It is easy to see that each solution of GMVVLI-1 is that of GMVVLI-2, but the converse may not be true in general. The aim of this paper is to derive the existence results of GMVVLI-1. It is well known that GMVVLI-1 and GMVVLI-2 are extensions of the vector variational inequality, which is a generalized form of a variational inequality, having applications in different areas of optimization, optimal control, operations research, economics equilibrium and free boundary value problems, see, for instance, [1-3] and the references therein. Now we give some definitions and lemmas needed for the proof of the existence results.

Definition 1.1 [4] Let $X$ be a subset of a topological space $E$. Then a set-valued mapping $F : X \to 2^E$ is called the KKM mapping if for each finite subset $\{x_1, x_2, \cdots, x_n\}$ of $X$, $\text{co} \{x_1, x_2, \cdots, x_n\} \subset \bigcup_{\iota=1}^n F(x_{\iota})$, where $\text{co} \{x_1, x_2, \cdots, x_n\}$ is the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

Definition 1.2 Let $H : X \times X \to Y$ be a mapping. $H$ is said to be affine in the second argument if, for each $x \in X$, $y_i \in X, (i = 1, 2, \cdots, n)$ with $\sum_{i=1}^n t_i = 1$ such that for each $y_i \in X, (i = 1, 2, \cdots, n)$, we have

$$H(x, \sum_{i=1}^n t_i y_i) = \sum_{i=1}^n t_i H(x, y_i).$$

Similarly, we can define the affinity of $H$ in the first argument.

II. EXISTENCE RESULTS

Theorem 2.1 Let $X$ be a compact convex subset of a Banach space $E$ and $Y$ a Banach space with convex cone $P$ such that $\text{int } P \neq \emptyset$ and $P \neq Y$. Let $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ be a continuous mapping. Suppose that:

(i) $M, S, T : X \to 2^{(E,Y)}$ are lower semicontinuous;

(ii) $H : X \times X \to Y$ and $\eta : X \times X \to Y$ are continuous in the second argument, respectively, and $H(x, x) = \eta(x, x) = 0$ for all $x \in X$;

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(iii) the multivalued mapping $w: X \rightarrow 2^Y$ defined by $W(x) = Y / \{ - \text{int } P \}$, has a closed graph in $X \times Y$;

(iv) for each $y \in X$,

$$B_y = \{ x \in X \colon \exists u \in M(y), v \in S(y), w \in T(y) $$

such that $\{ N(u, v, w), \eta(x, y) \} + H(x, y) < 0$ is convex. Then GMVVLI-1 is solvable.

**Proof.** Define a multivalued mapping $F: X \rightarrow 2^X$ by

$$F(x) = \left\{ y \in X : \{ N(u, v, w), \eta(x, y) \} + H(x, y) < 0 \right\} \quad \forall x \in X.$$ 

We first prove that $F$ is a KKM mapping. Suppose to the contrary, $F$ is not a KKM mapping. Then the convex hull of every finite subset $\{ x_1, x_2, \ldots, x_n \}$ of $X$ is not contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. Let $y \in \text{co} \{ x_1, x_2, \ldots, x_n \}$. Then $y = \sum_{i=1}^n \alpha_i x_i$ for some $\alpha_i \geq 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$ and $y \notin \bigcup_{i=1}^n F(x_i)$. Then we have for each $i \in \{1, 2, \ldots, n\}, \exists u \in M(y), v \in S(y), w \in T(y)$, such that

$$\{ N(u, v, w), \eta(x_i, y) \} + H(x_i, y) < 0.$$ 

Since by assumption (iv), $B_y$ is convex, then $\text{co} \{ x_1, x_2, \ldots, x_n \} \subseteq B_y$.

$$\{ N(u, v, w), \eta(\sum_{i=1}^n \alpha_i x_i) \} + H(\sum_{i=1}^n \alpha_i x_i, \sum_{i=1}^n \alpha_i x_i) \in -\text{int } P.$$ 

It follows from assumption (ii) that $0 \in -\text{int } P$, which contradicts $P \neq Y$. Therefore $F$ is a KKM mapping. Next we prove that for any $x \in X$, $F(x)$ is closed. Indeed, Let $\{ y_n \}$ be a sequence in $F(x)$ converging to $y^* \in X$. By the lower semicontinuity of $M, S, T, \forall (u^*, v^*, w^*) \in M(y^*) \times S(y^*) \times T(y^*)$, there exist $u_n \in M(y_n), v_n \in S(y_n), w_n \in T(y_n)$ for all $n$ such that $(u_n, v_n, w_n) \rightarrow (u^*, v^*, w^*) \in L(E, Y) \times L(E, Y) \times L(E, Y)$

Since $y_n \in F(x)$ for all $n$, we have

$$\{ N(u_n, v_n, w_n), \eta(x_n, y_n) \} + H(x, y_n) \not< 0,$$

which implies that

$$\{ N(u_n, v_n, w_n), \eta(x_n, y_n) \} + H(x, y_n) \in W(y_n).$$

It follows from assumption (ii) that $H$ and $\eta$ are continuous in the second argument, respectively, and note that $N$ is continuous, we have a closed graph in $X \times Y$ and $(u_n, v_n, w_n, y_n) \rightarrow (u^*, v^*, w^*, y^*)$, we get

$$\{ N(u_n, v_n, w_n), \eta(x, y_n) \} + H(x, y_n) \rightarrow \{ N(u^*, v^*, w^*), \eta(x, y^*) \} + H(x, y^*) \in W(y^*).$$

Then for some $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then for some $u \in M(y), v \in S(y), w \in T(y)$, we have

$$\{ N(u, v, w), \eta(x, y) \} + H(x, y) \in -\text{int } P.$$ 

This completes the proof.

**Corollary 2.1** Let $X$ be a compact convex subset of a Banach space $E$ and $Y$ a Banach space with convex cone $P$ such that, $\text{int } P \neq \emptyset$ and $P \neq Y$.

$$N: L(E, Y) \times L(E, Y) \times L(E, Y) \rightarrow L(E, Y)$$ 

be a continuous mapping. Assume that

(i) $M, S, T: X \rightarrow 2^{L(E,Y)}$ are lower semicontinuous;

(ii) $H: X \times X \rightarrow Y, \eta: X \times X \rightarrow Y$ are continuous in the second argument and affine in the first argument, respectively, and $H(x, x) = \eta(x, x) = 0$ for all $x \in X$;

(iii) the multivalued mapping $W: X \rightarrow 2^Y$ defined by $W(x) = Y / \{ - \text{int } P \}$, has a closed graph in $X \times Y$.

Then GMVVLI-1 is solvable.

**Proof** It is sufficient to prove that for each $y \in X$, the set

$$B_y = \{ x \in X : \exists u \in M(y), v \in S(y), w \in T(y) $$

such that $\{ N(u, v, w), \eta(x, y) \} + H(x, y) < 0$ is convex. For this, let $x_1, x_2 \in B_y$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then for some $u \in M(y), v \in S(y), w \in T(y)$, we have

$$\{ N(u, v, w), \eta(x_1, y) \} + H(x_1, y) \in -\text{int } P, \quad (1)$$
and
\[ \langle N(u,v,w),\eta(x_1,y) \rangle + H(x_1,y) \in -\text{int } P. \tag{2} \]

Nothing that \(-\text{int } P\) is a convex cone and multiplying (1) by \(\alpha\) and (2) by \(\beta\) and adding, we have
\[ \alpha\langle N(u,v,w),\eta(x_1,y) \rangle + \beta\langle N(u,v,w),\eta(x_2,y) \rangle + H(x_2,y) \in -(\alpha \text{int } P + \beta \text{int } P) = -\text{int } P. \tag{3} \]

It follows from the affinity of \(H\) and \(\eta\) in the first argument and (3) that
\[ \langle N(u,v,w),\alpha \eta(x_1,y) \rangle + \langle N(u,v,w),\beta \eta(x_2,y) \rangle + \alpha H(x_1,y) + \beta H(x_2,y) = \langle N(u,v,w),\eta(\alpha x_1 + \beta x_2,y) \rangle + H(\alpha x_1 + \beta x_2,y) \in -\text{int } P. \]

That is, \(\alpha x_1 + \beta x_2 \in B_y\), thus \(B_y\) is convex, completing the proof.

**Remark 2.1** Theorem 2.1 and corollary 2.1 extend the corresponding results of [5].

**Remark 2.2** If all the conditions of Theorem 2.1 and corollary 2.1 are satisfied, respectively, then GMVVLI-2 is solvable.

**References**


