

# Generalized Nonlinear Vector Variational-like Inequalities with set-valued mappings

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**Abstract**—In this paper, we introduce and study a class of generalized nonlinear vector variational-like inequalities. By utilizing maximal element theorem, we prove the existence of its solutions in the setting of locally convex topological vector space under certain conditions.

**Keywords**—Generalized nonlinear vector variational-like inequalities; Maximal element theorem; Upper semicontinuous; Diagonal convexity

## I. INTRODUCTION

Variational inequality theory has appeared as an effective and powerful tool to study and investigate a wide class of problems arising in pure and applied sciences including elasticity, optimization, economics, transportation, and structural analysis, see, for instance, [1-3] and the references therein. In 2010, Xiao, Fan and Qi [4] introduced and studied the following class of generalized nonlinear vector variational-like inequalities with set-valued mappings in locally convex topological vector space (locally convex space, in short) and obtained the existence of its solutions.

Let  $Z$  be a locally convex space,  $E$  be a Hausdorff topological vector space (t.v.s., in short). We denote by  $L(E, Z)$  the space of all continuous linear mappings from  $E$  into  $Z$  and by  $\langle l, x \rangle$  the evaluation of  $l \in L(E, Z)$  at  $x \in E$ . Let  $L(E, Z)$  be a space equipped with  $\sigma$ -topology. By the corollary of Schaefer ([5], P.80),  $L(E, Z)$  becomes a locally convex space. By Ding and Tarafdar [6], the bilinear mapping  $\langle \cdot, \cdot \rangle : L(E, Z) \times Z \rightarrow Z$  is continuous. Let  $K$  be a nonempty convex subset of a Hausdorff t.v.s.  $E$ ,  $C : K \rightarrow 2^Z$  be a set-valued mapping such that  $\text{int } C(x) \neq \emptyset$  for all  $x \in K$ ,  $\eta : K \times K \rightarrow E$  be a vector-valued mapping, let  $M, S, T : K \rightarrow 2^{L(E, Z)}$  and  $N : L(E, Z) \times L(E, Z) \times L(E, Z) \rightarrow 2^{L(E, Z)}$  and  $H : K \times K \rightarrow 2^Z$  be five set-valued mappings.

Find  $x \in K$  Such that  
 $\exists u \in M(x), v \in S(x)$  and  $w \in T(x)$  satisfying

$$\langle N(u, v, w), \eta(y, x) \rangle + H(x, y) \not\subseteq -\text{int } C(x), \forall y \in K.$$

In this paper, we introduce and consider the following generalized nonlinear vector variational-like inequalities with set-valued mappings (GNVVLIP, in short): Find  $x \in K$  such that

$$\exists u \in M(x), v \in S(x) \text{ and } w \in T(x) \text{ satisfying}$$

$$\langle N(u, v, w), \eta(y, g(x)) \rangle + H(g(x), y) \not\subseteq -\text{int } C(x), \forall y \in K,$$

where  $g : K \rightarrow K$  is a vector-valued mapping. Note that if  $g(x) = x$  for each  $x \in K$ , then the GNVVLIP collapses to the problem introduced and studied by Xiao, Fan and Qi in [4], which includes many known variational inequality problems as special cases, in detail, see [4] and the references therein. So the GNVVLIP is the most general and unifying one, which is also one of the main motivations of this paper.

## II. PRELIMINARIES

Let  $\text{int } A$  and  $\text{co}A$  denote the interior and convex hull of a set  $A$ , respectively. Let  $X, Y$  be two topological spaces,  $T : X \rightarrow 2^Y$  be a set-valued mapping and  $T^{-1}(y) = \{x \in X : y \in T(x)\}$ .  $T$  is said to be upper semicontinuous (u.s.c., in short) if for each  $x \in X$  and each open set  $V$  in  $Y$  with  $T(x) \subset V$ , there exists an open neighborhood  $U$  of  $x$  in  $X$  such that  $T(u) \subset V$  for each  $u \in U$ .  $T$  is closed if for any net  $\{x_\alpha\}$  in  $X$  such that  $x_\alpha \rightarrow x$  and any net  $\{y_\alpha\}$  in  $Y$  such that  $y_\alpha \rightarrow y$  and  $y_\alpha \in T(x_\alpha)$  for each  $\alpha$ , we have  $y \in T(x)$ .

**Definition 2.1** [7] Let  $K$  be a convex subset of a t.u.s.  $E$  and  $Z$  be a t.u.s.. Let  $C : K \rightarrow 2^Z$  be a set-valued mapping.

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Assume given any finite subset  $\Lambda = \{x_1, x_2, \dots, x_n\}$  of  $X$ , any  $x = \sum_{i=1}^n \alpha_i x_i$  with  $\alpha_i \geq 0$  for  $i=1, 2, \dots, n$ , and  $\sum_{i=1}^n \alpha_i = 1$ . Then

- (i) a single-valued mapping  $f : K \times K \rightarrow Z$  is said to be vector 0-diagonally convex in the second argument if  $\sum_{i=1}^n \alpha_i f(x, x_i) \notin -\text{int } C(x)$ .
- (ii) a set-valued mapping  $f : K \times K \rightarrow Z$  is said to be generalized vector 0-diagonally convex in the second argument if  $\sum_{i=1}^n \alpha_i f(x, x_i) \not\subseteq -\text{int } C(x)$ .

**Lemma 2.1** [8] Let  $X$  and  $Y$  be two topological spaces and  $T : X \rightarrow 2^Y$  an u.s.c. mapping with compact values. Suppose  $\{x_\alpha\}$  is a net in  $X$  such that  $x_\alpha \rightarrow x_0$ . If  $y_\alpha \in T(x_\alpha)$  for each  $\alpha$ , then there exist a  $y_0 \in T(x_0)$  and a subset  $\{y_\beta\}$  of  $\{y_\alpha\}$  such that  $y_\beta \rightarrow y_0$ .

**Lemma 2.2** [9] Let  $X$  and  $Y$  be two topological spaces. If  $T : X \rightarrow 2^Y$  an u.s.c. set-valued mapping with closed values, then  $T$  is closed.

**Lemma 2.3** [10] (Maximal Element Theorem) Let  $X$  be a nonempty convex subset of a t.v.s.  $E$ . Let  $S : X \rightarrow 2^X$  be a set-valued mapping satisfying the following conditions :

- (i)  $x \notin \text{co}S(x)$  and  $S^{-1}$  is open values;
- (ii) there exist a nonempty compact subset  $A$  of  $X$  and a nonempty compact convex subset  $B \subset X$  such that for all  $x \in X \setminus A$ , there exists  $z \in B$  such that  $x \in \text{int } S^{-1}(z)$ .

Then there exists  $x \in X$  such that  $S(x) = \emptyset$ .

### III. MAIN RESULTS

**Theorem 3.1** Let  $Z$  be a locally convex space,  $K$  be a nonempty convex subset of a Hausdorff t.v.s.  $E$ ,  $L(E, Z)$  be a space equipped with  $\sigma$ -topology,  $M, S, T : K \rightarrow 2^{L(E, Z)}$  be three u.s.c. set-valued mappings with nonempty compact values,  $g : K \rightarrow K$  be a vector-valued mapping. Suppose that the following conditions are satisfied:

- (i)  $H$  is generalized vector 0-diagonally convex in the second argument;
- (ii)  $\eta$  is continuous in both arguments and affine in the first argument with  $\eta(x, g(x)) = 0, \forall x \in K$ ;
- (iii)  $g$  is a continuous and affine mapping;
- (iv) for each  $y \in K$ ,

$$\langle N(\cdot, \cdot, \cdot), \eta(y, g(\cdot)) \rangle + H(g(\cdot), y) :$$

$$L(E, Z) \times L(E, Z) \times L(E, Z) \times K \times K \rightarrow 2^Z$$

is a upper semicontinuous set-valued mapping with compact values;

(v)  $C : K \rightarrow 2^Z$  is a set-valued mapping with convex values and the mapping  $G(x) := Z \setminus (-\text{int } C(x))$  from  $K$  into  $2^Z$  is upper semicontinuous;

(vi) there exist a nonempty compact subset  $A$  of  $X$  and a nonempty convex subset  $B \subset X$  such that for all  $x \in X \setminus A$ ,  $\exists \bar{y} \in B$  such that

$$\begin{aligned} & \langle N(u, v, w), \eta(\bar{y}, g(x)) \rangle + H(g(x), \bar{y}) \\ & \subseteq -\text{int } C(x), \forall u \in M(x), v \in S(x), w \in T(x). \end{aligned}$$

Then the GNVVLIP has a solution.

**Proof.** Define a set-valued  $P : K \rightarrow 2^K$  as follows: For each  $x \in K$ ,

$$\begin{aligned} P(x) &= \{y \in K : \langle N(u, v, w), \eta(y, g(x)) \rangle + H(G(x), y) \\ & \subseteq -\text{int } C(x), \forall u \in M(x), v \in S(x) \text{ and } w \in T(x)\}. \end{aligned}$$

First we show that  $x \notin \text{co}P(x)$  for all  $x \in K$ . Suppose to the contrary, there exists  $\bar{x} \in K$  such that  $\bar{x} \in \text{co}P(\bar{x})$ . Then there exist a finite set  $\{y_1, y_2, \dots, y_n\} \subset P(\bar{x})$  such that  $\bar{x} \in \text{co}\{y_1, y_2, \dots, y_n\}$ , we have  $\forall u \in M(x), v \in S(x)$  and  $w \in T(x)$ ,

$$\begin{aligned} & \langle N(u, v, w), \eta(y_i, g(\bar{x})) \rangle + H(g(\bar{x}), y_i) \\ & \subseteq -\text{int } C(\bar{x}), i=1, 2, \dots, n. \end{aligned}$$

Since  $\text{int } C(\bar{x})$  is a convex set and  $\eta$  is affine in the first argument, for  $\bar{x} = \sum_{i=1}^n t_i y_i \in K$ , where  $t_i \geq 0, i=1, 2, \dots, n$

with  $\sum_{i=1}^n t_i = 1$ , we obtain

$$\begin{aligned} & \left\langle N(u, v, w), \eta\left(\sum_{i=1}^n t_i y_i, g(\bar{x})\right) \right\rangle + \sum_{i=1}^n t_i \eta(g(\bar{x}), y_i) \\ & = \langle N(u, v, w), \eta(\bar{x}, g(\bar{x})) \rangle \\ & + \sum_{i=1}^n t_i \eta(g(\bar{x}), y_i) \subseteq -\text{int } C(\bar{x}). \end{aligned}$$

Since  $\eta(\bar{x}, g(\bar{x})) = 0$  by Condition (ii), we get  $\sum_{i=1}^n t_i \eta(g(\bar{x}), y_i) \subseteq -\text{int } C(x)$ , Which contradicts Condition (i). And so  $x \notin \text{co}P(x)$  for all  $x \in K$ . Next, we prove that for each  $y \in K$ ,  $P^{-1}(y)$  is a open set in  $K$ . For this, it is sufficient to prove that the complement

$$(P^{-1}(y))^c = \left\{ \begin{array}{l} x \in K : \\ \langle N(u, v, w), \eta(y, g(x)) \rangle + H(g(x), y) \\ \cap Z \setminus (-\text{int } C(x)) \neq \emptyset, \\ \forall u \in M(x), v \in S(x), w \in T(x) \end{array} \right\}$$

is closed in  $K$ . In fact, let  $\{x_\alpha\}$  be a net in  $(P^{-1}(y))^c$  such that  $x_\alpha \rightarrow \tilde{x}$ . Then there exist  $u_\alpha \in M(x_\alpha), v_\alpha \in S(x_\alpha),$

$w_\alpha \in T(x_\alpha)$  such that

$$\left\{ \begin{array}{l} \langle N(u_\alpha, v_\alpha, w_\alpha), \eta(y, g(x_\alpha)) \rangle + \eta(g(x_\alpha), y) \\ \cap Z \setminus (-\text{int } C(x_\alpha)) \neq \emptyset. \end{array} \right\}$$

Since  $M, S, T : K \rightarrow 2^{L(E, Z)}$  are three u.s.c. set-valued mappings with compact values, by Lemma 2.1  $\{u_\alpha\}, \{v_\alpha\}, \{w_\alpha\}$  have convergent subnets with limits, say,  $\tilde{u}, \tilde{v}$ , and,  $\tilde{w}, \tilde{u} \in M(\tilde{x}), \tilde{v} \in S(\tilde{x}), \tilde{w} \in T(\tilde{x})$ .

Without loss of generality, we may assume that  $u_\alpha \rightarrow \tilde{u}, v_\alpha \rightarrow \tilde{v}, w_\alpha \rightarrow \tilde{w}$ . Suppose that

$$\left\{ \begin{array}{l} z_\alpha \in \langle N(u_\alpha, v_\alpha, w_\alpha), \eta(y, g(x_\alpha)) \rangle + H(g(x_\alpha), y) \\ \cap Z \setminus \{-\text{int } C(x_\alpha)\}. \end{array} \right\}$$

Since  $\langle N(\cdot, \cdot, \cdot), \eta(y, g(\cdot)) \rangle + H(g(\cdot), y)$  is u.s.c. with compact values, by Lemma 2.1, there exists a

$\tilde{z} \in N(\tilde{u}, \tilde{v}, \tilde{w}), \eta(y, g(\tilde{x})) + H(g(\tilde{x}), y)$  and a subnet  $\{z_\beta\}$  of  $\{z_\alpha\}$  such that  $z_\beta \rightarrow \tilde{z}$ . On the other hand, since  $G(x) = Z \setminus \{-\text{int } C(x)\}$  is u.s.c. with closed values, which follows Lemma 2.2 that  $\tilde{z} \in Z \setminus \{-\text{int } C(\tilde{x})\}$ . hence

$$\langle N(\tilde{u}, \tilde{v}, \tilde{w}), \eta(y, G(\tilde{x})) \rangle + H(g(\tilde{x}), y) \cap Z \setminus \{-\text{int } C(\tilde{x})\} \neq \emptyset.$$

Thus  $(P^{-1}(y))^c$  is closed in  $K$ , meaning that  $P^{-1}(y)$  is open for each  $y \in K$ . Furthermore, by Condition (vi), we assert that  $\forall x \in K \setminus A, \exists \bar{y} \in B$  such that  $x \in \text{int } P^{-1}(\bar{y})$ . Hence,  $P$  satisfies all the conditions of Lemma 2.3. It follows that there exists an  $x \in K$  such that  $\exists u \in M(x), v \in S(x)$  and  $w \in T(x)$  satisfying

$$\langle N(u, v, w), \eta(y, g(x)) \rangle + H(g(x), y) \not\subseteq -\text{int } C(x), \forall y \in K.$$

This completes the proof.

If  $N, H$  reduce to be single-valued mappings, then we have the following result.

**Corollary 3.1** Let  $Z, K, L(E, Z), M, S, T, g$  be same as in Theorem 3.1. Let  $N : L(E, Z) \times L(E, Z) \times L(E, Z) \rightarrow L(E, Z)$

and  $H : K \times K \rightarrow Z$  be two single-valued mappings. Suppose that the following conditions are satisfied:

- (i)  $H$  is vector 0-diagonally convex in the second argument;
- (ii)  $\eta : K \times K \rightarrow E$  is continuous in both arguments and affine in the first argument with  $\eta(x, g(x)) = 0, \forall x \in K$ ;
- (iii)  $g$  is continuous and affine;
- (iv)  $C : K \rightarrow 2^Z$  is a set-valued mapping with convex values and  $Z \setminus (-\text{int } C(x))$  is an upper semicontinuous mapping;
- (vi) there exist a nonempty compact subset  $A$  of  $K$  and a nonempty compact convex subset  $B$  of  $K$  such that  $\forall x \in K \setminus A, \exists \bar{y} \in B$  such that

$$\langle N(u, v, w), \eta(\bar{y}, g(x)) \rangle + H(g(x), \bar{y}) \in -\text{int } C(x), \forall u \in M(x), v \in S(x) \text{ and } w \in T(x).$$

Then there exists an  $x \in K$  Such that  $\exists u \in M(x), v \in S(x)$  and,  $w \in T(x)$  Satisfying

$$\langle N(u, v, w), \eta(y, g(x)) \rangle + H(g(x), y) \not\subseteq -\text{int } C(x), \forall y \in K.$$

**Remark 3.1** Theorem 3.1 and corollary 3.1 extend the corresponding results of [4].

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