Simulation of Poisson process

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Abstract. The paper mainly introduces the definition and characteristic numbers of Poisson process and also gives two simulation methods and operation steps on Poisson process. In this paper, we mainly introduce the idea of generating the interval. We also give an example and use Matlab to simulate the process. At last, we make a simple analysis about the simulation.

Introduction

Poisson process is the basic process of independent increments which cumulate the number of occurrences of random events. For example, as time increases, the number of cumulative call which a telephone exchange has received is a Poisson process. Poisson process plays an important role in theory and application of stochastic processes, especially in service systems and queuing theory.

Definition of Poisson process

Definition 1[1]: We define stochastic process $\{N(t), t \geq 0\}$ is a counting process. If $N(t)$ represents the total number of "event A" which had occurred up to tine t and $N(t)$ satisfies the following conditions:

a) $N(t) \geq 0$;

b) $N(t)$ is positive integer

c) If $s < t$, $N(s) \leq N(t)$;

d) When $s < t, N(t) - N(s)$equals number of "event A" which have occurred in $(s, t]$

Definition 2[2]: If we get a counting process $\{X(t), t \geq 0\}$ which value are non-negative integers and it satisfies the following conditions:

a) X(0) = 0;

b) X(t) is an independent increments process;

c) In any interval which length is t, the total number of "event A" is a Poisson distribution and its parameter $\lambda t > 0$,for any of $s, t \geq 0$,

$$P\{X(t+s) - X(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n = 0,1 \ldots$$

From the condition c) we can know that Poisson process is a process with stationary increment and $E[X(t)] = \lambda t$.For $\lambda = \frac{E[X(t)]}{t}$ stands for the average number of "event A" in unit time. Therefore λ is called rate or strength of this process.

From the definition 2, we can see that to judge whether the counting process is a Poisson process or

not. We must prove it satisfies condition a),b)and c).The condition a) only explains that the counting of event A starts from t=0;Condition b) usually can be verified from the situation we have known. Condition a) and b) are easy to verify but the verification of condition c) is very difficult. Therefore we give an equivalent definition of Poisson process.

Definition 3[2]: If we get a counting process $\{X(t), t \geq 0\}$ which value are non-negative integers and it satisfies the following conditions:

a) X(0) = 0;
b) X(t) is an independent increments process with stationary increment;
c) X(t) satisfies:

$$\mathrm{P}\{X(t+h) - X(t) = 1\} = \lambda h + o(h)$$

$$\mathrm{P}\{X(t+h) - X(t) \geq 2\} = o(h)$$

Then we call $\{X(t), t \geq 0\}$ is a Poisson process and its parameter isλ.

Characteristic Numbers of Poisson Process

Characteristic Number: For any of $\mathrm{t} > 0, \mathrm{N(t)} \sim \Psi(\lambda \mathrm{t})$

a) mean value function:$\mathrm{m(t)} = \mathrm{E}\{N(t)\} = \lambda t$
b) variance function:$\mathrm{D}\{N(t)\} = \lambda t$
c) covariance function:$\mathrm{B(s,t)} = \lambda \min(\mathrm{s,t})$;
d) correlation function:$\mathrm{R(s,t)} = \lambda \min(\mathrm{s,t}) + \lambda^2 st$

One-dimensional correlation function:

$$\varphi(\mathrm{u}) = \mathrm{E}\left[e^{iuN(t)}\right] = \sum_{k=0}^{\infty} e^{iuk} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \sum_{k=0}^{\infty} \frac{(\lambda t e^{iu})^k}{k!} e^{-\lambda t} = e^{\lambda t e^{iu}} e^{-\lambda t} = e^{\lambda t (e^{iu} - 1)} \quad (1)$$

Stochastic Simulation of Poisson Process

Method 1:

Theorem 1[3]: Assume $\{N(t), t \geq 0\}$ is a Poisson process and its parameter isλ. $\{W_n, n = 1,2, \dots\}$ is a sequence of waiting time. Then $W_n \sim \Gamma(n, \lambda)$ and the probability density function as follows:

$$\mathrm{f(t)} = \begin{cases} \frac{\lambda^n}{\Gamma_n} t^{n-1} e^{-\lambda t}, t \geq 0 \\ 0, t < 0 \end{cases} \quad (2)$$

Theorem 2[3]: Assume $\{N(t), t \geq 0\}$ is a poisson process and its parameter isλ. $\{T_n, n = 1,2, \dots\}$ is an interval sequence. Then $\{T_n, n = 1,2, \dots\}$ are independent and identically distributed random variables and they obey exponential distribution which parameter is λ.

From theorem 1 and 2 we can know that if we want to generate a Poisson process, we should generate the interval between two successive events of the process first.

If$\mathrm{U} \sim \mathrm{U}(0,1)$, $\Delta \mathrm{t} = -\frac{1}{\lambda} \ln(1 - U)$, then we can gain the distribution function of $\Delta \mathrm{t}$ F(T):

$$\mathrm{F(T)} = \mathrm{P}\{\Delta t \leq t\} = P\left\{-\frac{1}{\lambda} \ln(1 - U) \leq t\right\} = P\left\{U \leq 1 - e^{-\lambda t}\right\} = 1 - e^{-\lambda t} \quad (3)$$

Therefore $\Delta \mathrm{t}$ obeys exponential distribution which parameter is λ .So we can use the following algorithm to simulate Poisson process.

t represents time, $\Delta \mathrm{t}$ represents the interval, I represents the number of events occur in time t, S(I) represents the time of event I.

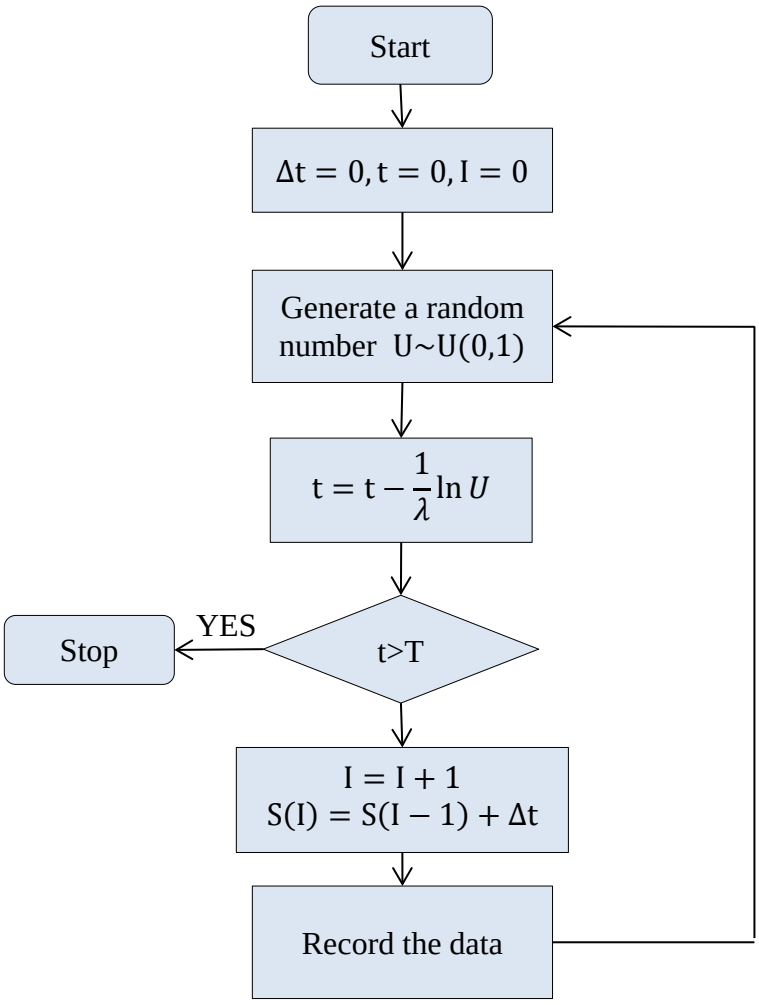

Fig. 1: Flow chart of simulation by method 1.

Method 2:

We know the probability of an event occurring within the time intervalΔt :
$P(N(\Delta t) = 1) = \lambda \Delta t + o(\Delta t)$.Then we can get another method to simulate the state of Poisson process which parameter is λ before the time T.

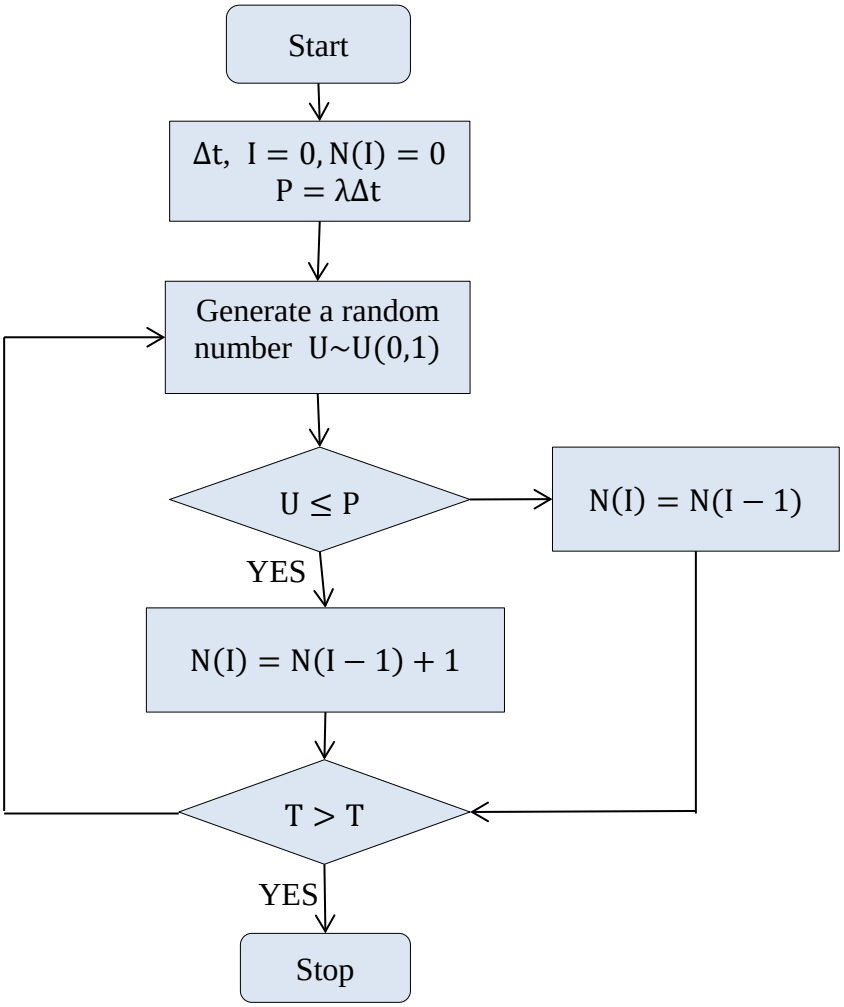

Fig. 2: Flow chart of simulation by method 2.

An Example of Simulation

Assume that particles reach a counter with average speed 4/min and this process is a Poisson process. $N(t)$ represents the number of particles reaching the counter in time of $[0, t)$.We do 100 experiments.

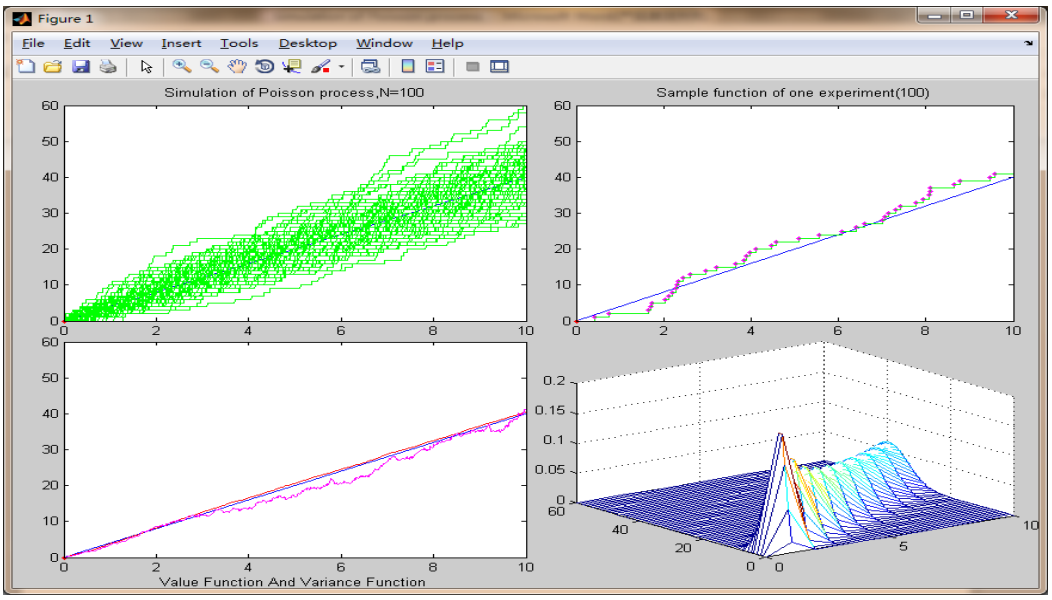

Fig. 3: The results of simulation by method 1.

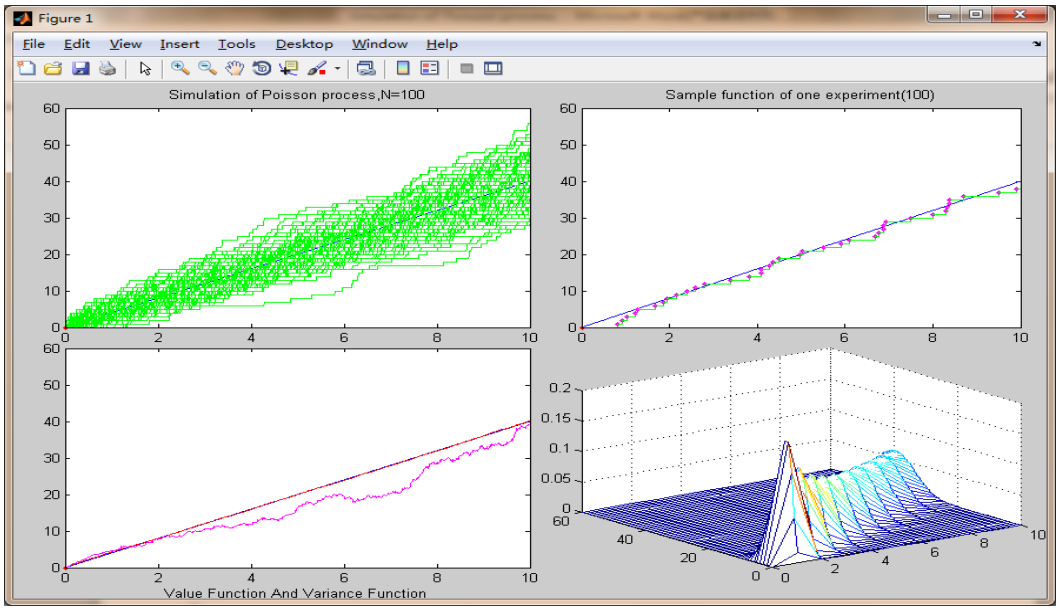

Fig. 4: The results of simulation by method 2.

In Fig. 3 and Fig. 4, the diagram in the top right displays the state of each simulation. The abscissa represents time (from 0 to 10 minutes). Y-axis representsN(t). The blue line shows the theory of mean value function. The diagram in the top left displays the state of 100 experiments The diagram in the bottom left show the mean value function (red line) and the variance function (pink line) of the 100 experiments. The diagram in the bottom right shows the probability density function.
With the increase of the number of experiments, we can find that the curve of mean value function getting close to the theoretical value. There is no significant difference of the two methods.

Acknowledgements

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