Computer Applications to Exact solutions of AKNS Hierarchy with Variable Coefficients

Sheng Zhang^{1, a}, Bin Cai^{1, b}

¹School of Mathematics and Physics, Bohai University, Jinzhou 121013, PR China

^aszhangchina@126.com, ^b65470641@qq.com

Keywords: Computer symbolic computation; Exact solution; AKNS hierarchy

Abstract. It is well known that computer has been widely used in almost all fields. In this paper, Hirota's bilinear method with computer symbolic computation is extended to a new AKNS hierarchy with variable coefficients. As a result, one-soliton solutions and two-soliton solutions are obtained. Through the analysis of the obtained one-soliton solutions and two-soliton solutions, a uniform formulae of n -soliton solutions are derived. It is shown from computer running that the obtained one-soliton solutions and two-soliton solutions satisfy the AKNS hierarchy.

Introduction

In the past several decades, some powerful symbolic computation systems like *Mathematica* or *Maple* have become indispensable auxiliary tools for solving nonlinear partial differential equations (PDEs) [1-8]. Finding exact soliton solutions of nonlinear PDEs plays an important role in the study of nonlinear physical phenomena and has gradually developed into a significant direction in nonlinear science. Some effective methods for constructing multi-soliton solutions have been proposed, such as those in [9-11]. Among them, Hirota's bilinear method [11] is a purely algebraic method, the process of which is fairly simple and convenient for computer operation. Since the Hirota's bilinear method was proposed, the method has received extensively applications [12-15]. However there are very few studies on generalizing the method for a whole hierarchy of nonlinear PDEs. In this article, we will extend the Hirota's bilinear method to a new AKNS hierarchy with variable coefficients

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_{m+1}} = \alpha(t_{m+1}) L^m \begin{pmatrix} q \\ r \end{pmatrix}_x, \quad t_0 = x, \quad m = 0, 1, 2, \dots \quad (1)$$

for constructing its multi-soliton solutions, here the recursion operator is employed as

$$L = \sigma \partial + 2 \begin{pmatrix} q \\ -r \end{pmatrix} \partial^{-1} (r, q), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \frac{1}{2} \left(\int_{-\infty}^x dx - \int_x^{\infty} dx \right), \quad (2)$$

when $\alpha(t_{m+1}) = 1$, Eq. (1) becomes the constant-coefficient AKNS hierarchy [14].

Multi-soliton solutions

To begin with, we rewrite Eq. (1) as

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_{m+1}} = \frac{\alpha(t_{m+1})}{\alpha(t_m)} L \begin{pmatrix} q \\ r \end{pmatrix}_{t_m}. \quad (3)$$

Then from Eq. (3) we have

$$q_{t_{m+1}} = \frac{\alpha(t_{m+1})}{\alpha(t_m)} \left[-q_{t_m, x} + 2q \partial^{-1} (qr)_{t_m} \right], \quad (4)$$

$$r_{t_{m+1}} = \frac{\alpha(t_{m+1})}{\alpha(t_m)} \left[r_{t_m, x} - 2r \partial^{-1} (qr)_{t_m} \right]. \quad (5)$$

Further taking the following transformation

$$q = \frac{g}{f}, \quad r = \frac{h}{f}, \quad f = f(x, t_1, t_2, \dots), \quad g = g(x, t_1, t_2, \dots), \quad h = h(x, t_1, t_2, \dots), \quad (5)$$

we get the bilinear forms of the AKNS hierarchy (1):

$$D_{t_{m+1}} g \cdot f = \frac{\alpha(t_{m+1})}{\alpha(t_m)} \left\{ -D_{t_{m+1}} D_x g \cdot f + fg \partial^{-1} \left[\frac{1}{f^2} (D_x^2 f \cdot f + 2gh) \right] \right\}_{t_m}, \quad (6)$$

$$D_{t_{m+1}} h \cdot f = \frac{\alpha(t_{m+1})}{\alpha(t_m)} \left\{ D_{t_{m+1}} D_x h \cdot f - fh \partial^{-1} \left[\frac{1}{f^2} (D_x^2 f \cdot f + 2gh) \right] \right\}_{t_m}. \quad (7)$$

If we set

$$D_x^2 f \cdot f + 2g \cdot h = 0, \quad (8)$$

then Eqs. (6) and (7) are reduced to

$$\left[D_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} D_{t_m} D_x \right] g \cdot f = 0, \quad (9)$$

$$\left[D_{t_{m+1}} - \frac{\alpha(t_{m+1})}{\alpha(t_m)} D_{t_m} D_x \right] h \cdot f = 0. \quad (10)$$

In what follows, based on the reduced bilinear forms (9) and (10) we construct multi-soliton solutions of the AKNS hierarchy (1).

To construct one-soliton solutions, we suppose that

$$f = 1 + \varepsilon^2 f^{(2)} + \varepsilon^4 f^{(4)} + \dots + \varepsilon^{2j} f^{(2j)} + \dots, \quad (11)$$

$$g = \varepsilon^1 g^{(1)} + \varepsilon^3 g^{(3)} + \dots + \varepsilon^{2j+1} g^{(2j+1)} + \dots, \quad (12)$$

$$h = \varepsilon^1 h^{(1)} + \varepsilon^3 h^{(3)} + \dots + \varepsilon^{2j+1} h^{(2j+1)} + \dots. \quad (13)$$

With the help of symbolic computation system *Mathematica* we substitute Eqs. (11)-(13) into Eqs. (8)-(10) and then collect the coefficients of the same order of ε , a system of differential equations are obtained as following

$$\partial_{t_{m+1}} g^{(1)} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} \partial_{t_m} \partial_x g^{(1)} = 0, \quad (14)$$

$$\partial_{t_{m+1}} h^{(1)} - \frac{\alpha(t_{m+1})}{\alpha(t_m)} \partial_{t_m} \partial_x h^{(1)} = 0, \quad (15)$$

$$\partial_x^2 f^{(2)} + g^{(1)} h^{(1)} = 0, \quad (16)$$

$$\left[\partial_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} \partial_{t_m} \partial_x \right] g^{(3)} = - \left[D_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} D_{t_m} D_x \right] g^{(1)} f^{(2)}, \quad (17)$$

$$\left[\partial_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} \partial_{t_m} \partial_x \right] h^{(3)} = - \left[D_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} D_{t_m} D_x \right] h^{(1)} f^{(2)}, \quad (18)$$

$$2\partial_x^2 f^{(4)} = -D_x^2 f^{(2)} f^{(2)} - 2(g^{(1)} h^{(3)} + g^{(3)} h^{(1)}), \quad (19)$$

$$\left[\partial_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} \partial_{t_m} \partial_x \right] g^{(5)} = - \left[D_{t_{m+1}} + \frac{\alpha(t_{m+1})}{\alpha(t_m)} D_{t_m} D_x \right] (g^{(1)} f^{(4)} + g^{(3)} f^{(2)}), \quad (20)$$

$$\left[\partial_{t_{m+1}} - \frac{\alpha(t_{m+1})}{\alpha(t_m)} \partial_{t_m} \partial_x \right] h^{(5)} = - \left[D_{t_{m+1}} - \frac{\alpha(t_{m+1})}{\alpha(t_m)} D_{t_m} D_x \right] [h^{(1)} f^{(4)} + h^{(3)} f^{(2)}], \quad (21)$$

$$2\partial_x^2 f^{(6)} = -D_x^2 f^{(2)} f^{(4)} - 2(g^{(1)} h^{(5)} + g^{(3)} h^{(3)} + g^{(5)} h^{(1)}), \quad (22)$$

$\vdots$

and so forth. If let

$$g^{(1)} = e^{\varepsilon_1}, \quad \xi_1 = k_1 x + \sum_{m=0}^{\infty} w_{1,m+1} \int \alpha(t_{m+1}) dt_{m+1}, \quad (23)$$

$$h^{(1)} = e^{\eta_1}, \quad \eta_1 = l_1 x + \sum_{m=0}^{\infty} \sigma_{1,m+1} \int \alpha(t_{m+1}) dt_{m+1}, \quad (24)$$

then we can obtain

$$w_{1,m+1} = -k_1 w_{1,m}, \quad w_{1,1} = k_1, \quad (25)$$

$$\sigma_{1,m+1} = l_1 \sigma_{1,m}, \quad \sigma_{1,1} = l_1. \quad (26)$$

Taking into consideration of Eq. (16), we suppose that

$$f^{(2)} = e^{\varepsilon_1 + \eta_1 + \theta_{13}}, \quad (27)$$

where θ_{13} is a constant to be determined. From Eqs. (23), (24) and (27) we obtain

$$e^{\theta_{13}} = -\frac{1}{(k_1 + k_2)^2}. \quad (28)$$

Substituting Eqs. (23)-(28) into Eqs. (17)-(19), we can see that when

$$g^{(3)} = h^{(3)} = f^{(4)} = \dots = 0, \quad (29)$$

Eq. (20) and those behind all hold. We then write

$$f_1 = 1 + e^{\varepsilon_1 + \eta_1 + \theta_{13}}, \quad g_1 = e^{\varepsilon_1}, \quad h_1 = e^{\eta_1}. \quad (30)$$

and obtain one-soliton solutions of the AKNS hierarchy (1):

$$q = \frac{e^{\varepsilon_1}}{1 + e^{\varepsilon_1 + \eta_1 + \theta_{13}}}, \quad r = \frac{e^{\eta_1}}{1 + e^{\varepsilon_1 + \eta_1 + \theta_{13}}}. \quad (31)$$

For the two-soliton solutions, we suppose that

$$g^{(1)} = e^{\varepsilon_1} + e^{\varepsilon_2}, \quad \xi_2 = k_2 x + \sum_{m=0}^{\infty} w_{2,m+1} \int \alpha(t_{m+1}) dt_{m+1}, \quad (32)$$

$$h^{(1)} = e^{\eta_1} + e^{\eta_2}, \quad \eta_2 = l_2 x + \sum_{m=0}^{\infty} \sigma_{2,m+1} \int \alpha(t_{m+1}) dt_{m+1}, \quad (33)$$

and substitute them into Eqs. (14) and (15). We then obtain

$$w_{2,m+1} = -k_2 w_{2,m}, \quad w_{2,1} = k_2, \quad (34)$$

$$\sigma_{2,m+1} = l_2 \sigma_{2,m}, \quad \sigma_{2,1} = l_2. \quad (35)$$

In view of Eq. (16), we further let

$$f^{(2)} = e^{\varepsilon_1 + \eta_1 + \theta_{13}} + e^{\varepsilon_1 + \eta_2 + \theta_{14}} + e^{\varepsilon_2 + \eta_1 + \theta_{23}} + e^{\varepsilon_2 + \eta_2 + \theta_{24}}, \quad (36)$$

where θ_{13} , θ_{14} , θ_{23} and θ_{24} are constants to be determined later. Substituting Eq. (36) into Eq. (16) yields

$$e^{\theta_{14}} = -\frac{1}{(k_1 + l_2)^2}, \quad e^{\theta_{23}} = -\frac{1}{(k_2 + l_1)^2}, \quad e^{\theta_{24}} = -\frac{1}{(k_2 + l_2)^2}. \quad (37)$$

Taking into consideration of Eq. (17), we select

$$g^{(3)} = e^{\varepsilon_1 + \varepsilon_2 + \eta_1 + \theta_{12} + \theta_{13} + \theta_{23}} + e^{\varepsilon_1 + \varepsilon_2 + \eta_2 + \theta_{12} + \theta_{14} + \theta_{24}}, \quad (38)$$

where θ_{12} , θ_{13} , θ_{14} , θ_{23} and θ_{24} are undetermined constants. We then obtain from Eqs. (17), (32) and (36)

$$e^{\theta_{12}} = -(k_1 - k_2)^2. \quad (39)$$

In view of Eqs. (18) and (19), we set

$$h^{(3)} = e^{\varepsilon_1 + \eta_1 + \eta_2 + \theta_{13} + \theta_{14} + \theta_{34}} + e^{\varepsilon_2 + \eta_1 + \eta_2 + \theta_{23} + \theta_{24} + \theta_{34}}, \quad (40)$$

$$f^{(4)} = e^{\varepsilon_1 + \varepsilon_2 + \eta_1 + \eta_2 + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{23} + \theta_{24} + \theta_{34}}, \quad (41)$$

and then obtain

$$e^{\theta_{34}} = -(l_1 - l_2)^2. \quad (42)$$

Substituting Eqs. (40)-(42) into Eq. (20), we can see that if

$$g^{(5)} = h^{(5)} = f^{(6)} = \dots = 0. \quad (43)$$

then Eq. (21) and those behind all hold. In this case, we obtain two-soliton solutions of AKNS hierarchy (1):

$$q_2 = \frac{g_2}{f_2}, \quad r_2 = \frac{h_2}{f_2}, \quad (44)$$

where

$$g_2 = e^{\varepsilon_1} + e^{\varepsilon_2} + e^{\varepsilon_1 + \varepsilon_2 + \eta_1 + \theta_{12} + \theta_{13} + \theta_{23}} + e^{\varepsilon_1 + \varepsilon_2 + \eta_2 + \theta_{12} + \theta_{14} + \theta_{24}}, \quad (45)$$

$$h_2 = e^{\varepsilon_1} + e^{\varepsilon_2} + e^{\varepsilon_1 + \eta_1 + \eta_2 + \theta_{13} + \theta_{14} + \theta_{34}} + e^{\varepsilon_2 + \eta_1 + \eta_2 + \theta_{23} + \theta_{24} + \theta_{34}}, \quad (46)$$

$$f_2 = 1 + e^{\varepsilon_1 + \eta_1 + \theta_{13}} + e^{\varepsilon_1 + \eta_2 + \theta_{14}} + e^{\varepsilon_2 + \eta_1 + \theta_{23}} + e^{\varepsilon_2 + \eta_2 + \theta_{24}} + e^{\varepsilon_1 + \varepsilon_2 + \eta_1 + \eta_2 + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{23} + \theta_{24} + \theta_{34}}, \quad (47)$$

$$e^{\theta_{12}} = -(k_1 - k_2)^2, \quad e^{\theta_{34}} = -(l_1 - l_2)^2, \quad e^{\theta_{i(j+2)}} = -\frac{1}{(k_i + l_j)^2}, \quad i, j = 1, 2. \quad (48)$$

Through the analysis of the obtained one-soliton solutions (31) and two-soliton solutions (44), we can construct multi-soliton solutions. If we take

$$f_n = \sum_{\mu=0,1} B_1(\mu) e^{\sum_{i=1}^{2n} \mu_i \xi_i + \sum_{1 \leq i < j}^{2n} \mu_i \mu_j \theta_{ij}}, \quad (49)$$

$$g_n = \sum_{\mu=0,1} B_2(\mu) e^{\sum_{i=1}^{2n} \mu_i \xi_i + \sum_{1 \leq i < j}^{2n} \mu_i \mu_j \theta_{ij}}, \quad (50)$$

$$h_n = \sum_{\mu=0,1} B_3(\mu) e^{\sum_{i=1}^{2n} \mu_i \xi_i + \sum_{1 \leq i < j}^{2n} \mu_i \mu_j \theta_{ij}}, \quad (51)$$

$$\xi_j = k_j x + \sum_{m=0}^{\infty} w_{1,m+1} \int \alpha(t_{m+1}) dt_{m+1}, \quad w_{j,m+1} = -k_j w_{j,m}, \quad w_{j,1} = k_j, \quad (52)$$

$$\eta_j = l_j x + \sum_{m=0}^{\infty} \sigma_{1,m+1} \int \alpha(t_{m+1}) dt_{m+1}, \quad \sigma_{j,m+1} = l_j \sigma_{j,m}, \quad \sigma_{j,1} = l_j, \quad j = 1, 2, \dots, n, \quad (53)$$

$$e^{\theta_{ij}} = -(k_i - k_j)^2, \quad e^{\theta_{(i+n)(j+n)}} = -(l_i - l_j)^2, \quad i < j = 2, 3, \dots, n, \quad (54)$$

$$e^{\theta_{i(j+n)}} = -\frac{1}{(k_i + l_j)^2}, \quad i, j = 1, 2, \dots, n, \quad (55)$$

then a uniform formula of n -soliton solutions of the AKNS hierarchy (1) are obtained as following:

$$q_n = \frac{g_n}{f_n}, \quad r_n = \frac{h_n}{f_n}, \quad (56)$$

where the summation $\sum_{\mu=0,1}$ refers to all possible combination of each $\mu=0,1$ for $i=1,2,\dots,n$, $B_1(\mu)$, $B_2(\mu)$ and $B_3(\mu)$ denote that when we select all the possible combinations $\mu_j=0,1$ ($j=1,2,\dots,2n$) the following conditions hold, respectively

$$\sum_{j=1}^n \mu_j = \sum_{j=1}^n \mu_{n+j}, \quad \sum_{j=1}^n \mu_j = \sum_{j=1}^n \mu_{n+j} + 1, \quad \sum_{j=1}^n \mu_j + 1 = \sum_{j=1}^n \mu_{n+j}. \quad (57)$$

To the best of our knowledge, the obtained one-soliton solutions (31), two-soliton solutions (44) and n -soliton solutions (56) are new, they have not been reported in literature. With the help of symbolic computation system *Mathematica*, both the one-soliton solutions (31) and the two-soliton solutions (44) have been tested by substituting them back into the AKNS hierarchy (1). The computer running affirms the validity of solutions (31) and (44).

Acknowledgements

This work was supported by the PhD Start-up Funds of Liaoning Province of China (20141137) and Bohai University (bsqd2013025), the Natural Science Foundation of Educational Committee of Liaoning Province of China (L2012404), the Liaoning BaiQianWan Talents Program (2013921055) and the Natural Science Foundation of China (11371071).

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