Efficient Three-point Iterative Methods for Solving Nonlinear Equations

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Abstract. In this paper, we present a new family of three-point Newton type iterative methods for solving nonlinear equations. The order of convergence of the new family without memory is eight requiring the evaluations of three functions and one first-order derivative in per full iteration. Numerical examples are demonstrated to confirm theoretical results.

Introduction

Solving nonlinear equations is one of the most important problems in scientific computation. Since 1960's, many iterative methods have been proposed for solving nonlinear equations, see [1-5] and the references therein. Specifically, Wang and Zhang in [1] developed the following iterative method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{\lambda f(x_n) + f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{2\lambda f(x_n) + f'(x_n)}(1 + 2s_n + \tau s_n^2), \end{cases} \quad (1)$$

where $\lambda, \tau \in R$ and $s_n = f(y_n) / f(x_n)$.

This paper is organized as follows. In Section 2, based on Wu's method, we derive a new family of Newton type iterative methods without memory for solving nonlinear equations. We prove that the order of convergence of the new methods without memory is eight requiring the evaluations of three functions and one first-order derivative in per full iteration. Numerical examples are given in Section 4 to confirm theoretical results.

The New Methods without Memory

Now, we consider the following iteration scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{\lambda f(x_n) + f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{2\lambda f(x_n) + f'(x_n)}(1 + 2s_n + \tau s_n^2), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \end{cases} \quad (2)$$

where $\lambda, \tau \in R$. To derive a scheme with higher efficiency index, we approximate $f'(z_n)$ by using the Taylor expansion and divided difference. By using the Taylor expansion, $f'(z_n)$ in (2) can be approximated by

$$f'(z_n) \approx f[z_n, y_n] + \frac{1}{2} f''(y_n)(z_n - y_n) + \frac{1}{3} f'''(y_n)(z_n - y_n)^2. \quad (3)$$

In order to avoid the computation of the second-order derivative and the third-order derivative,

we approximate $f''(y_n)$ and $f'''(y_n)$ as follows:

$$f''(y_n) \approx 2f[z_n, x_n, x_n], \quad (4)$$

$$f'''(y_n) \approx 6f[z_n, x_n, x_n, y_n]. \quad (5)$$

We approximate $(z_n - y_n)^2$ as follows:

$$(z_n - y_n)^2 \approx (z_n - y_n)(z_n - x_n). \quad (6)$$

Substituting (4)-(6) into (3), we obtain an approximation of $f'(z_n)$:

$$f'(z_n) \approx f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n) + 2f[z_n, x_n, x_n, y_n](z_n - y_n)(z_n - x_n). \quad (7)$$

Therefore, a new scheme is derived as follows:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{\lambda f(x_n) + f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{2\lambda f(x_n) + f'(x_n)} \left(1 + 2\frac{f(y_n)}{f(x_n)} + \tau \left(\frac{f(y_n)}{f(x_n)}\right)^2\right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n) + 2f[z_n, x_n, x_n, y_n](z_n - y_n)(z_n - x_n)}. \end{cases} \quad (8)$$

Using the Taylor series and symbolic computation in the programming package Mathematica, we can find the order of convergence and the asymptotic error constant (AEC) of the three-step methods (8). For simplicity, we sometimes omit the iteration index n and write e instead of e_n . The approximation x_{n+1} to the root a will be denoted by $\hat{x}$. Regarding (8), let us define the errors $e = x - a$, $d = y - a$, $p = z - a$, $e1 = \hat{x} - a$.

First, let us introduce the following abbreviations used in the program.

$$ck = f^{(k)}(a) / (k! f'(a)), \quad e = x - a, \quad d = y - a, \quad p = z - a, \quad e1 = \hat{x} - a, \quad L = \lambda, \quad T = \tau$$

$$fx = f(x), \quad fy = f(y), \quad fz = f(z), \quad dfx = f'(x), \quad fla = f'(a), \quad fxy = f[x, y], \quad s = fy/fx$$

$$fyz = f[y, z], \quad fxz = f[x, z], \quad fxxxy = f[x, x, y], \quad fzxx = f[z, x, x], \quad fzxxxy = f[z, x, x, y].$$

Program (written in Mathematica 7)

$$\begin{aligned} fx &= fla * (e + c2 * e^2 + c3 * e^3 + c4 * e^4 + c5 * e^5 + c6 * e^6 + c7 * e^7 + c8 * e^8); & dfx &= D[fx, e]; \\ d &= Series[e - fx / (L * fx + dfx), \{e, 0, 8\}]; & fy &= Series[fla * (d + c2 * d^2 + c3 * d^3 + c4 * d^4), \{e, 0, 8\}]; & s &= fy / fx; \\ p &= Series[d - fy / (2 * L * fx + dfx) * (1 + 2 * s + T * s^2), \{e, 0, 8\}]; & fz &= Series[fla * (p + c2 * p^2), \{e, 0, 8\}]; \\ fxy &= Series[(fy - fx) / (d - e), \{e, 0, 8\}]; & fyz &= Series[(fy - fz) / (d - p), \{e, 0, 8\}]; \\ fxz &= Series[(fx - fz) / (e - p), \{e, 0, 8\}]; \\ fxxxy &= Series[(dfx - fxy) / (e - d), \{e, 0, 8\}]; & fzxx &= Series[(dfx - fxz) / (e - p), \{e, 0, 8\}]; \\ fzxxxy &= Series[(fzxx - fxxxy) / (p - d), \{e, 0, 8\}]; \\ e1 &= Series[p - fz / (fyz + fzxx * (p - d) + 2 * fzxxxy * (p - e) * (p - d)), \{e, 0, 8\}] // Simplify \\ Out[e1] &= (c2 + L)^2 (c3 + c2^2 (-5 + T) + L^2 (-2 + T) + c2L (-7 + 2T)) (-c4 - c3L \\ &\quad + c2^3 (-5 + T) + c2L^2 (-2 + T) + c2^2L (-7 + 2T)) e^8 + O[e]^9. \end{aligned} \quad (9)$$

The output (9) of the above program means that the family of three-step methods defined by (8) is of eighth-order. Altogether, we can state the following theorem.

Theorem1. Let I be an open interval and $a \in I$ be a simple zero point of a sufficiently differentiable function $f : I \rightarrow R$. Then the new method defined by (8) is of eighth order.

Numerical Results

Now, the new methods (8) are employed to solve some nonlinear equations and compared with the following optimal eighth-order iterative methods.

Wang-Liu method (WLM, see [2]):

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{2f[x_n, y_n] - f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + f[x_n, x_n, y_n](y_n - z_n)}. \end{cases} \quad (10)$$

Bi-Wu-Ren method (BRM, see [3]):

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - (1 + 2\frac{f(y_n)}{f(x_n)} + 5(\frac{f(y_n)}{f(x_n)})^2) \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n) + 3f(z_n)}{f(x_n) + f(z_n)} \frac{f(z_n)}{f[z_n, y_n] + f[x_n, x_n, z_n](z_n - y_n)}. \end{cases} \quad (11)$$

The absolute errors $|x_k - a|$ in the first three iterations are given in Table 1, where a is the exact root computed with 2400 significant digits. The computational order of convergence ρ is defined by [4]:

$$\rho \approx \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}. \quad (12)$$

Following test functions are used:

$$f_1(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, \quad a \approx -1.2076478271309189, \quad x_0 = -1.3, \quad \lambda = 1,$$

$$f_2(x) = x^5 + x^4 + 4x^2 - 15, \quad a \approx 1.3474280989683050, \quad x_0 = 1.6, \quad \lambda = -1.$$

TABLE 1 NUMERICAL RESULTS OF THE DIFFERENT METHODS

Function	Methods	$ x_1 - a $	$ x_2 - a $	$ x_3 - a $	ρ
$f_1(x)$	WLM	0.11867e-7	0.59442e-63	0.23564e-505	8.0000000
	BRM	0.40513e-6	0.32351e-48	0.53484e-385	8.0000001
	(8), $\tau = 5$	0.11251e-8	0.80741e-72	0.56799e-577	8.0000000
$f_2(x)$	WLM	0.37439e-5	0.15738e-43	0.15343e-350	7.9999999
	BRM	0.84179e-4	0.62964e-31	0.61512e-248	8.0000456
	(8), $\tau = 5$	0.11392e-7	0.11281e-66	0.10431e-538	8.0000000

Conclusions

The numerical results shown in Table 1 are in concordance with the theory developed in this paper. The results obtained with our optimal eighth-order methods without memory are better than the other optimal eighth-order methods without memory.

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