

2-Rainbow Domination of the Circulant Graph $C(n; \{1, 3\})$

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Abstract—Let G be a graph where for each vertex, a subset of a set of k colors is assigned. If for each vertex to which an empty set is assigned, its neighborhood contains all k colors, then such an assignment is called a k -rainbow dominating function of G . The corresponding invariant $\gamma_{rk}(G)$, which is the minimum sum of the cardinalities of the subsets assigned by a k -rainbow dominating function of G , is called the k -rainbow domination number of G . In this paper, we study the 2-rainbow domination number of the Circulant graph $C(n; \{1, 3\})$, and we show that $\gamma_{r2}(C(n; \{1, 3\})) = 2 \left\lfloor \frac{n}{5} \right\rfloor + \alpha$,

where $\alpha = 0$ for $n \equiv 0 \pmod{5}$, $\alpha = 1$ for $n \equiv 1, 2 \pmod{5}$ and $\alpha = 2$ for $n \equiv 3, 4 \pmod{5}$.

Keywords—the Circulant graph; rainbow domination; 2-rainbow domination number

I. INTRODUCTION

Rainbow domination is one of the most active research fields in recent years in the theory of graph domination. The study of rainbow domination in graphs is active in many fields of optimization theory, the design and analysis of communication networks, network search, and pattern recognition, with plenty of practical significance.

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2, \dots, k\}$; that is, $f: V(G) \rightarrow P(\{1, 2, \dots, k\})$. If $\bigcup_{u \in N(v)} f(u) = \{1, 2, \dots, k\}$ for each vertex $v \in V(G)$ with $f(v) = \emptyset$, then f is called a k -rainbow domination function (k RDF) of G . The weight, $w(f)$, of a function f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a k RDF is called the k -rainbow domination number of G , which is denoted by $\gamma_{rk}(G)$. Clearly when $k = 1$ this concept coincides with the ordinary domination.

The concept of rainbow domination was introduced by

Brešar B, Henning M A and Rall D F [1,2]. In [2], it was observed that rainbow domination of a graph G coincides with the ordinary domination of the Cartesian product of G with the complete graph, in particular $\gamma_{rk}(G) = \gamma(G \square K_k)$ for any graph G . The relationship between rainbow domination and paired-domination of Cartesian products of graphs was also studied. In addition, a linear-time algorithm for determining a minimum weight 2-rainbow dominating function of an arbitrary tree was presented. In the language of domination of Cartesian products, B. Hartnell and D.F. Rall [4] obtained several observations about rainbow domination, for instance:

$$\min\{|G|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G) \quad (1)$$

for any $k \geq 2$ and any graph G . The attempt in to characterize graphs with $\gamma(G) = \gamma_{r2}(G)$ was inspired by the following famous open problem [10].

Vizing's Conjecture. For any graphs G and H , $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.

One of the related problems posed in [5] is to find classes of graphs that achieve the equality. There it was shown that $\gamma(G \square H) = \gamma(G)\gamma(H)$, if G is any graph with $\gamma(G) = \gamma_{r2}(G)$ and H is a so-called generalized comb.

In [3], Brešar B and Sumenjak T K showed that the problem of deciding if a graph has a 2-rainbow dominating function of a given weight is NP-complete. In addition, some bounds and exact results for several standard classes of graphs were proved, namely, paths, cycles, suns and generalized Petersen graphs. For relatively prime integers $1 \leq k < n$ with $n \geq 3$, the generalized Petersen graph $P(n, k)$ is defined as follows. Let V and V' be two disjoint cycles of length n , say $V = u_0 u_1 \dots u_{n-1} u_0$ and $V' = v_0 v_k v_{2k} \dots v_{(n-1)k} v_0$, with indices taken modulo n . The

graph $P(n, k)$ is obtained by adding the edges $u_i v_i$ for $0 \leq i \leq n-1$. The graph $P(5, 2)$ is the well-known Petersen graph. Brešar B and Sumenjak T K showed that $\left\lceil \frac{4n}{5} \right\rceil \leq \gamma_{r_2}(P(n, 2)) \leq \left\lceil \frac{4n}{5} \right\rceil + \beta$, where $\beta = 0$ for $n \equiv 3, 9 \pmod{10}$ and $\beta = 1$ for $n \equiv 1, 5, 7 \pmod{10}$, and $\left\lceil \frac{4n}{5} \right\rceil \leq \gamma_{r_2}(P(n, k)) \leq n$ for relatively prime numbers n and k . Yunjian Wu and Nader Jafari Rad[13] also studied bounds on the 2-rainbow domination number of graphs.

In recent years, the research on domination in graphs has made great progress. Xueliang Fu[6,11] and Yuansheng Yang[12] studied the domination number, and also presented the redomination number concept. Chunling Tong[9], Meiqin Luo[8] and Guangjun Xu[7] further studied the generalized Petersen graphs $P(n, 2)$ and $P(n, 3)$, and the following results were proved.

Theorem 1.1. Chunling Tong[9] proved this Theorem.

$$\gamma_{r_2}(P(n, 2)) = \begin{cases} \left\lceil \frac{4n}{5} \right\rceil, & n \equiv 0, 3, 4, 9 \pmod{10}, \\ \left\lceil \frac{4n}{5} \right\rceil + 1, & n \equiv 1, 2, 5, 6, 7, 8 \pmod{10}. \end{cases} \quad (2)$$

Theorem 1.2. Guangjun Xu[7] proved this Theorem ($n \geq 13$).

$$\gamma_{r_2}(P(n, 3)) = \begin{cases} n - \left\lfloor \frac{8}{n} \right\rfloor, & n \equiv 0, 2, 4, 5, 6, 7, 13, 14, 15 \pmod{16}, \\ n - \left\lfloor \frac{8}{n} \right\rfloor + 1, & n \equiv 1, 3, 8, 9, 10, 11, 12 \pmod{16}. \end{cases} \quad (3)$$

For $1 \leq j < k \leq \frac{n}{2}$, the Circulant graph $G = C(n; \{j, k\})$ is the graph on n vertices whose vertex and edge sets are as follows:

$$V(C(n; \{j, k\})) = V(G) = \{v_i : 0 \leq i \leq n-1\}, \quad (4)$$

$$E(C(n; \{j, k\})) = E(G) = \{v_i v_{i+j}, v_i v_{i+k} : 0 \leq i \leq n-1\} \quad (\text{subscripts are taken module } n). \quad (5)$$

The Circulant graph $C(7; \{1, 3\})$ is shown in Figure 1.

In this paper, we prove the following result.

Theorem 1.3. For $n \geq 7$, let $m = \left\lfloor \frac{n}{5} \right\rfloor$ and $n = 5m + \alpha$. we have

$$\gamma_{r_2}(C(n; \{1, 3\})) = \begin{cases} 2m, & \alpha = 0, \\ 2m + 1, & \alpha = 1, 2, \\ 2m + 2, & \alpha = 3, 4. \end{cases} \quad (6)$$

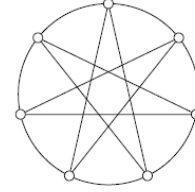


FIGURE 1. THE CIRCULANT GRAPH $C(7; \{1, 3\})$

II. 2-RAINBOW DOMINATION NUMBER OF THE CIRCULANT GRAPH $C(n; \{1, 3\})$

In this section, we prove Theorem 1.3. For convenience, let 0, 1, 2, 3 stand for, respectively, the values $\emptyset, \{1\}, \{2\}, \{1, 2\}$ taken by a 2-rainbow domination function. We first prove the upper bound.

Let:

$$V_0 = \{v \in V(G) : f(v) = \emptyset\}, \quad (7)$$

$$V_1 = \{v \in V(G) : f(v) = \{1\} \text{ or } f(v) = \{2\}\}, \quad (8)$$

$$V_2 = \{v \in V(G) : f(v) = \{1, 2\}\}, \quad (9)$$

$$V'(x, i) = \{v_{x+j} : 0 \leq j \leq i-1\}, \quad (10)$$

$$N(v) = \{u \in V(G) : uv \in E(G)\}, \quad (11)$$

$$N[v] = N(v) \cup \{v\}, \quad (12)$$

$$rd(S) = \sum_{v \in S} rd(v), \quad (13)$$

$$rd(N_e[v \cup u]) = \sum_{v \in N_e[v \cup u]} rd(v), \quad (14)$$

$$rd(V'(x, i)) = \sum_{v \in V'(x, i)} rd(v), \quad (15)$$

$$f(v_0, v_1, \dots, v_{n-1}) = (f(v_0), f(v_1), \dots, f(v_{n-1})). \quad (16)$$

For a vertex v , let $p(v)$ be open neighborhood rainbow domination weight of the vertex v , and $q(v)$ be its own rainbow domination weight of the vertex v , respectively, that is to say:

$$\text{For } v_a \in V_0, \text{ let } p(v_a) = 0, \quad q(v_a) = 0. \quad (17)$$

$$\text{For } v_b \in V_1, \text{ let } p(v_b) = 0.5, \quad q(v_b) = 1. \quad (18)$$

$$\text{For } v_c \in V_2, \text{ let } p(v_c) = 1, \quad q(v_c) = 2. \quad (19)$$

Definition 1. The number of repeat rainbow domination (NRRD) of a vertex v , denoted by $rd(v)$, is defined by:

$$rd(v) = \sum_{u \in N(v)} p(u) + q(v) - 1. \quad (20)$$

Definition 2. The effective rainbow dominating vertices (ERDV) of S , where S is either a single vertex of $V_1 \cup V_2$, or $S = v \cup u$ where $f(v) = \{1\}$ and $f(u) = \{2\}$, is the set $N_e[S]$ defined as follows. If $v \in V_1$, $N_e[v] = \{v\}$; if $v \in V_2$, then $N_e[v] = N[v]$; if $f(v) = \{1\}$, $f(u) = \{2\}$, then $N_e[v \cup u] = (N(v) \cap N(u)) \cup \{v, u\}$. Note that $N_e[S]$ consists of those vertices that are 2-rainbow dominated by S .

Definition 3. Rainbow dominating subset: Rainbow dominating subset of all vertices are the rainbow domination vertex. If any one vertex v belongs to the rainbow dominating subset, then $f(v) = \{1\}$, $f(v) = \{2\}$, or $f(v) = \{1, 2\}$.

Lemma 2.1. For $n \geq 7$, let $m = \lfloor \frac{n}{5} \rfloor$ and $n = 5m + \alpha$, we have

$$\gamma_{r2}(C(n; \{1, 3\})) \leq \begin{cases} 2m, & \alpha = 0, \\ 2m + 1, & \alpha = 1, 2, \\ 2m + 2, & \alpha = 3, 4. \end{cases} \quad (21)$$

Proof. Now we prove the upper bound of Theorem 1.3. For a 2RDF $f: V(G) \rightarrow P(\{1, 2\})$, let:

$$f(v_0, v_1, \dots, v_{n-1}) = \begin{cases} 01020 \dots 01020, & \alpha = 0, \\ 01020 \dots 010201, & \alpha = 1, \\ 01020 \dots 0102010, & \alpha = 2, \\ 01020 \dots 01020102, & \alpha = 3, \\ 01020 \dots 010200102, & \alpha = 4. \end{cases} \quad (22)$$

Clearly, the function f is a 2RDF of $C(n; \{1, 3\})$ with

$$w(f) = \begin{cases} 2m, & \alpha = 0, \\ 2m + 1, & \alpha = 1, 2, \\ 2m + 2, & \alpha = 3, 4. \end{cases} \quad (23)$$

So

$$\gamma_{r2}(C(n; \{1, 3\})) \leq \begin{cases} 2m, & \alpha = 0, \\ 2m + 1, & \alpha = 1, 2, \\ 2m + 2, & \alpha = 3, 4. \end{cases} \quad (21)$$

Lemma 2.2. For the Circulant graph $C(8; \{1, 3\})$, if $v \in V_1(f(v) = \{1\})$, $u \in V_1(f(u) = \{2\})$, and $|N_e[v \cup u]| = 6$, then $rd(N_e[v \cup u]) > 1$.

Proof. Without loss of generality, let $f(v_1) = \{1\}$, $f(v_3) = \{2\}$, then $|N_e[v_1 \cup v_3]| = 6$. Because $v_5 \notin N[v_1 \cup v_3]$, $v_7 \notin N[v_1 \cup v_3]$, according to the definition of 2-RDF, there is at least one rainbow dominating vertex and its $w(f) = 2$. Suppose $f(v_6) = \{1, 2\}$, then $rd(N_e[v_1 \cup v_3]) = 2 \times 1 + 2 = 4 > 1$ (as shown in Figure 2(1)) (the other cases ($f(v_2) = \{1, 2\}$, $f(v_4) = \{1, 2\}$ and $f(v_8) = \{1, 2\}$ are similar to this case). Suppose $f(v_5) = \{1\}$, $f(v_7) = \{2\}$, then $rd(N_e[v_1 \cup v_3]) = 2 \times 4 \times 0.5 = 4 > 1$ (as shown in Figure 2(2)) (the other cases (such as $f(v_4) = \{1\}$, $f(v_6) = \{2\}$) are similar to this case).

Lemma 2.3. For the Circulant graph $C(n; \{1, 3\})$ ($n \geq 7$ and $n \neq 8$), if $f(v_i) = \{1\}$, $f(v_{i+j}) = \{2\}$ ($1 \leq j \leq 6$), and $|N_e[v_i \cup v_{i+j}]| \leq 6$, then $rd(V'(i-1, 5)) \geq 1$, or $rd(V'(i-1, 10)) \geq 2$, or $rd(V'(i-1, 5)) = 0.5$ and $rd(V'(i-6, 5)) \geq 2.5$.

Proof. There are 6 cases.

Case 1. $f(v_i) = \{1\}$, $f(v_{i+1}) = \{2\}$, then $rd(v_i) = 0.5$, $rd(v_{i+1}) = 0.5$. Therefore, $rd(V'(i-1, 5)) \geq rd(v_i) + rd(v_{i+1}) = 0.5 + 0.5 = 1 \geq 1$ (as shown in Figure 3).

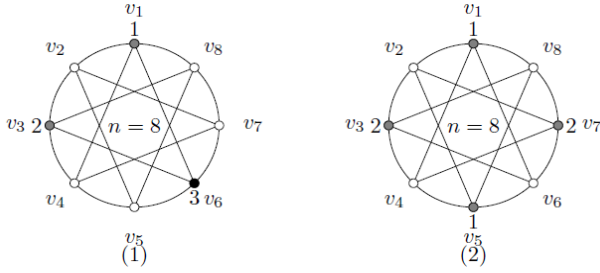


FIGURE II. CASE FOR THE PROOF OF LEMMA 2.2

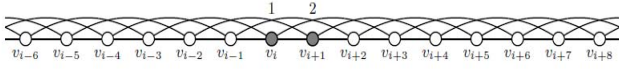


FIGURE III. $f(v_i) = \{1\}, f(v_{i+1}) = \{2\}$

Case 2. $f(v_i) = \{1\}$, $f(v_{i+2}) = \{2\}$ (as shown in Figure 4(1)), there are 3 cases.

Case 2.1. $v_{i+5} \in V_0$, there are 3 cases.

Case 2.1.1. $f(v_{i+4}) = \{1\}$, then $rd(v_{i+1}) = rd(v_{i+3}) = 0.5$.

Therefore, $rd(V'(i-1, 5)) \geq rd(v_{i+1}) + rd(v_{i+3}) = 0.5 + 0.5 = 1 \geq 1$ (as shown in Figure 4(2)).

Case 2.1.2. $f(v_{i+6}) = \{1\}$, then $rd(v_{i+3}) = 0.5$.

Considering the vertex v_{i+4} , we have $f(v_{i+7}) = \{1, 2\}$, then $rd(v_{i+6}) = 1$, $rd(v_{i+7}) = 1.5$. Therefore, $rd(V'(i-1, 10)) \geq rd(v_{i+3}) + rd(v_{i+6}) + rd(v_{i+7}) = 0.5 + 1 + 1.5 = 3 > 2$ (as shown in Figure 4(3)).

Case 2.1.3. $f(v_{i+8}) = \{1\}$. Considering vertices v_{i+4} and v_{i+6} , we have $f(v_{i+7}) = \{1, 2\}$, then $rd(v_{i+7}) = 1.5$,

$rd(v_{i+8}) = 1$. Therefore, $rd(V'(i-1, 10)) \geq rd(v_{i+7}) + rd(v_{i+8}) = 1.5 + 1 = 2.5 > 2$ (as shown in Figure 4(4)).

Case 2.2. $v_{i+5} \in V_1$, suppose $f(v_{i+5}) = \{1\}$, then $rd(v_{i+2}) = 0.5$. Considering vertex v_{i+4} , then $f(v_{i+7}) = \{2\}$. If $v_{i-4} \notin V_0$, or $v_{i-3} \notin V_0$, or $v_{i-2} \notin V_0$, then $rd(V'(i-1, 5)) \geq 1$. If $v_{i-4} \in V_0$ and $v_{i-3} \in V_0$ and $v_{i-2} \in V_0$, then $f(v_{i-5}) = \{1, 2\}$, $f(v_{i-6}) = \{2\}$, $rd(v_{i-6}) = 1$, $rd(v_{i-5}) = 1.5$. Therefore, $rd(V'(i-1, 5)) = 0.5$ and $rd(V'(i-6, 5)) \geq rd(v_{i-6}) + rd(v_{i-5}) = 1 + 1.5 = 2.5 \geq 2.5$ (as shown in Figure 4(5)).

Case 2.3. $v_{i+5} \in V_2$, then $rd(v_{i+2}) = 1$. Therefore, $rd(V'(i-1, 5)) \geq rd(v_{i+2}) = 1 \geq 1$ (as shown in Figure 4(6)).

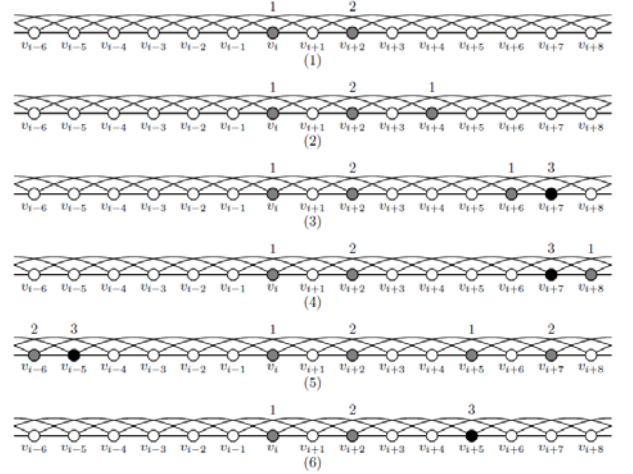


FIGURE IV. $f(v_i) = \{1\}, f(v_{i+2}) = \{2\}$

Case 3. $f(v_i) = \{1\}$, $f(v_{i+3}) = \{2\}$, then $rd(v_i) = 0.5$, $rd(v_{i+3}) = 0.5$. Therefore, $rd(V'(i-1, 5)) \geq rd(v_i) + rd(v_{i+3}) = 0.5 + 0.5 = 1 \geq 1$ (as shown in Figure 5).

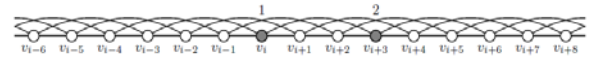


FIGURE V. $f(v_i) = \{1\}, f(v_{i+3}) = \{2\}$

Case 4. $f(v_i) = \{1\}$, $f(v_{i+4}) = \{2\}$ (as shown in Figure 6(1)). Considering the vertex v_{i+2} , there are 3 cases.

Case 4.1. $v_{i+2} \in V_0$, there are 3 cases.

Case 4.1.1. If $v_{i-1} \in V_1$, then $rd(v_{i-1})=0.5$, $rd(v_i)=0.5$, $rd(V'(i-1,5)) \geq rd(v_{i-1}) + rd(v_i) = 0.5 + 0.5 = 1 \geq 1$ (suppose $f(v_{i-1}) = \{1\}$, as shown in Figure 6(2)); if $v_{i+1} \in V_1$, then $rd(v_i)=0.5$, $rd(v_{i+1})=1$, $rd(V'(i-1,5)) \geq rd(v_i) + rd(v_{i+1}) = 0.5 + 1 = 1.5 > 1$; if $v_{i+3} \in V_1$, then $rd(v_i)=0.5$, $rd(v_{i+3})=1$, $rd(V'(i-1,5)) \geq rd(v_i) + rd(v_{i+3}) = 0.5 + 1 = 1.5 > 1$.

Case 4.1.2. If $v_{i-1} \in V_2$, then $rd(v_{i-1})=1.5$, $rd(v_i)=1$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1}) + rd(v_i) = 1.5 + 1 = 2.5 > 1$ (as shown in Figure 6(3)). If $v_{i+1} \in V_2$, or $v_{i+3} \in V_2$, similarly available, $rd(V'(i-1,5)) > 1$.

Case 4.1.3. If $v_{i+5} \in V_2$, then $rd(v_{i+4})=1$, $rd(v_{i+5})=1.5$. Therefore, $rd(V'(i-1,10)) \geq rd(v_{i+4}) + rd(v_{i+5}) = 1 + 1.5 = 2.5 > 2$ (as shown in Figure 6(4)).

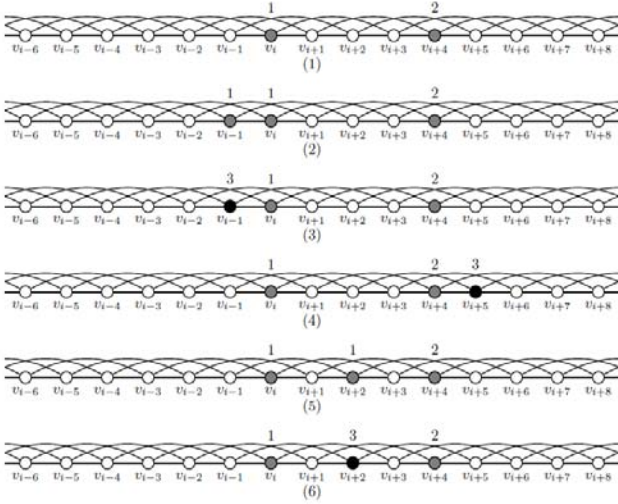


FIGURE VI. $f(v_i) = \{1\}, f(v_{i+4}) = \{2\}$

Case 4.2. $v_{i+2} \in V_1$. If $f(v_{i+2}) = 2$, it is the same as case 2; if $f(v_{i+2}) = 1$, then $rd(v_{i+1}) = rd(v_{i+3}) = 0.5$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i+1}) + rd(v_{i+3}) = 0.5 + 0.5 = 1 \geq 1$ (as shown in Figure 6(5)).

Case 4.3. $v_{i+2} \in V_2$, then $rd(v_{i-1})=0.5$, $rd(v_{i+1})=rd(v_{i+2})=rd(v_{i+3})=1$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1}) +$

$rd(v_{i+1}) + rd(v_{i+2}) + rd(v_{i+3}) = 0.5 + 1 + 1 + 1 = 3.5 > 1$ (as shown in Figure 6(6)).

Case 5. $f(v_i) = \{1\}, f(v_{i+5}) = \{2\}$ (as shown in Figure 7(1)). Considering the vertex v_{i+1} , if $f(v_{i+2}) = \{2\}$, it is the same as case 2; if $f(v_{i+4}) = \{2\}$, it is the same as case 4. So we only need to consider $f(v_{i-2}) = \{2\}$. Considering the vertex v_{i+2} , there are 3 cases.

Case 5.1. $v_{i+2} \in V_0$, there are 3 cases.

Case 5.1.1. If $f(v_{i-1}) = \{1\}$, then $rd(v_{i-1})=1$, $rd(v_i)=0.5$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1}) + rd(v_i) = 1 + 0.5 = 1.5 > 1$ (as shown in Figure 7(2)).

Case 5.1.2. If $f(v_{i+1}) = \{1\}$, then $rd(v_i)=0.5$, $rd(v_{i+1})=1$. Therefore, $rd(V'(i-1,5)) \geq rd(v_i) + rd(v_{i+1}) = 0.5 + 1 = 1.5 > 1$ (as shown in Figure 7(3)).

Case 5.1.3. If $f(v_{i+3}) = \{1\}$, then $rd(v_i)=0.5$, $rd(v_{i+3})=0.5$. Therefore, $rd(V'(i-1,5)) \geq rd(v_i) + rd(v_{i+3}) = 0.5 + 0.5 = 1 \geq 1$ (as shown in Figure 7(4)).

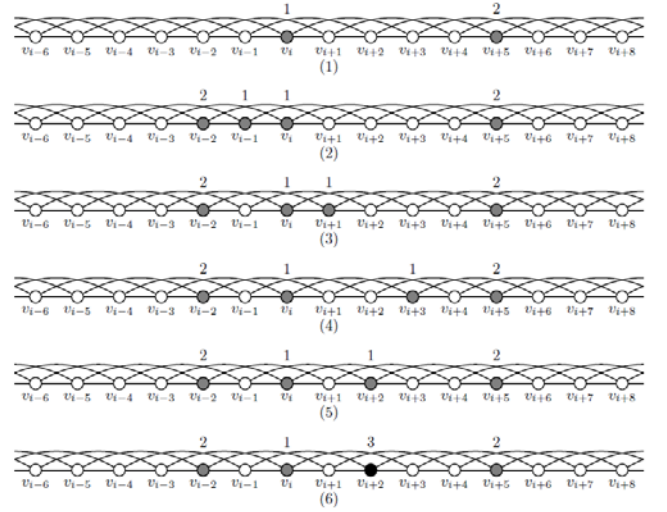


FIGURE VII. $f(v_i) = \{1\}, f(v_{i+5}) = \{2\}$

Case 5.2. $v_{i+2} \in V_1$. If $f(v_{i+2}) = \{2\}$, it is the same as case 2; if $f(v_{i+2}) = \{1\}$, then $rd(v_{i-1}) = rd(v_{i+1}) = rd(v_{i+2}) = 0.5$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1}) + rd(v_{i+1}) + rd(v_{i+2}) = 0.5 + 0.5 + 0.5 = 1.5 > 1$ (as shown in Figure 7(5)).

Case 5.3. $v_{i+2} \in V_2$, then $rd(v_{i-1})=rd(v_{i+1})=1$, $rd(v_{i+2})=1.5$, $rd(v_{i+3})=0.5$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1})+rd(v_{i+1})+rd(v_{i+2})+rd(v_{i+3})=1+1+1.5+0.5=4>1$ (as shown in Figure 7(6)).

Case 6. $f(v_i) = \{1\}$, $f(v_{i+6}) = \{2\}$ (as shown in Figure 8(1)), there are 3 cases.

Case 6.1. $v_{i+2} \in V_0$, there are 3 cases.

Case 6.1.1. If $v_{i-1} \in V_1$, then $rd(v_{i-1})=0.5$, $rd(v_i)=0.5$, $rd(V'(i-1,5)) \geq rd(v_{i-1})+rd(v_i)=0.5+0.5=1 \geq 1$ (suppose $f(v_{i-1}) = \{1\}$, as shown in Figure 8(2)); if $v_{i+1} \in V_1$, then $rd(v_i)=0.5$, $rd(v_{i+1})=0.5$, $rd(V'(i-1,5)) \geq rd(v_i)+rd(v_{i+1})=0.5+0.5=1 \geq 1$; if $v_{i+3} \in V_1$, then $rd(v_i)=0.5$, $rd(v_{i+3})=1$, $rd(V'(i-1,5)) \geq rd(v_i)+rd(v_{i+3})=0.5+1=1.5>1$.

Case 6.1.2. If $v_{i-1} \in V_2$, then $rd(v_{i-1})=1.5$, $rd(v_i)=1$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1})+rd(v_i)=1.5+1=2.5>1$ (as shown in Figure 8(3)). If $v_{i+1} \in V_2$, or $v_{i+3} \in V_2$, similarly available, $rd(V'(i-1,5))>1$.

Case 6.1.3. If $v_{i+5} \in V_2$, then $rd(v_{i+5})=1.5$, $rd(v_{i+6})=1$. Therefore, $rd(V'(i-1,10)) \geq rd(v_{i+5})+rd(v_{i+6})=1.5+1=2.5>2$ (as shown in Figure 8(4)).

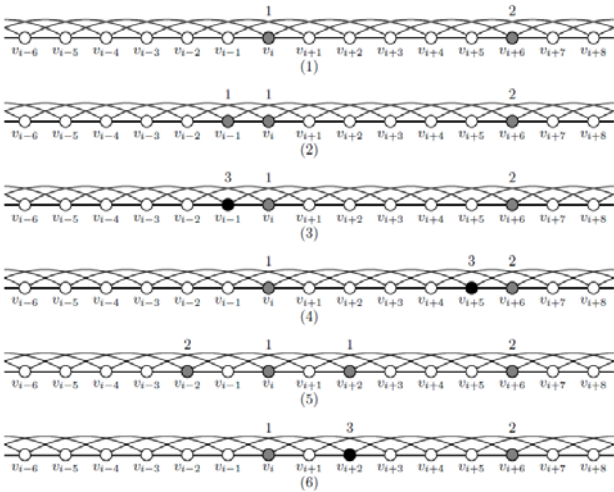


FIGURE VIII. $f(v_i) = \{1\}$, $f(v_{i+6}) = \{2\}$

Case 6.2. $v_{i+2} \in V_1$. If $f(v_{i+2}) = \{2\}$, it is the same as case 2. If $f(v_{i+2}) = \{1\}$, then $rd(v_{i+3})=0.5$. Considering the vertex v_{i+1} , If $f(v_{i+4}) = \{2\}$, it is the same as case 4. If $f(v_{i-2}) = \{2\}$, then $rd(v_{i-1})=rd(v_{i+1})=0.5$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1})+rd(v_{i+1})+rd(v_{i+3})=0.5+0.5+0.5=1.5>1$ (as shown in Figure 8(5)).

Case 6.3. $v_{i+2} \in V_2$, then $rd(v_{i-1})=rd(v_{i+1})=0.5$, $rd(v_{i+2})=rd(v_{i+3})=1$. Therefore, $rd(V'(i-1,5)) \geq rd(v_{i-1})+rd(v_{i+1})+rd(v_{i+2})+rd(v_{i+3})=0.5+0.5+1+1=3>1$ (as shown in Figure 8(6)).

For any a rainbow dominating subset S , let

$$|N_d[S]| = \sum_{v \in S} |N_d[v]| = p(S) + q(S) = \sum_{v \in S} p(v) + \sum_{v \in S} q(v), \quad (24)$$

$$w(f(S)) = \sum_{v \in S} |f(v)|. \quad (25)$$

Lemma 2.4. For the Circulant graph $C(n; \{1,3\})$, if the rainbow dominating subset S' satisfies $w(f(S'))=2$, we have $|N_d[S']|=6$.

Proof. According to the definition of 2-rainbow domination function and $w(f(S'))=2$, then the rainbow dominating subset S' is either a single vertex $S'=v$ where $v \in V_2$, or $S'=v \cup u$ where $v \in V_1$ and $u \in V_1$. There are 2 cases.

Case 1. Suppose $v \in V_2$, then $|N_d[S']| = p(S') + q(S') = \sum_{v \in S'} p(v) + \sum_{v \in S'} q(v) = 4 \times 1 + 2 = 6$.

Case 2. Suppose $v \in V_1$ and $u \in V_1$, there are 3 cases.

Case 2.1. Suppose $f(v)=f(u)=\{1\}$, then $|N_d[S']| = p(S') + q(S') = \sum_{v \in S'} p(v) + \sum_{v \in S'} q(v) = 2 \times 4 \times 0.5 + 1 + 1 = 6$.

Case 2.2. Suppose $f(v)=f(u)=\{2\}$, then $|N_d[S']| = p(S') + q(S') = \sum_{v \in S'} p(v) + \sum_{v \in S'} q(v) = 2 \times 4 \times 0.5 + 1 + 1 = 6$.

Case 2.3. Suppose $f(v) = \{1\}$ and $f(u) = \{2\}$, then
 $|N_d[S']| = p(S') + q(S') = \sum_{v \in S'} p(v) + \sum_{v \in S'} q(v)$
 $= 2 \times 4 \times 0.5 + 1 + 1 = 6$.

We can now obtain the lower bound for Theorem 1.3.

Lemma 2.5. For $n \geq 7$, let $m = \lfloor \frac{n}{5} \rfloor$ and $n = 5m + \alpha$, we have

$$\gamma_{r_2}(C(n; \{1,3\})) \geq \begin{cases} 2m, & \alpha = 0, \\ 2m + 1, & \alpha = 1, 2, \\ 2m + 2, & \alpha = 3, 4. \end{cases} \quad (26)$$

Proof. Let f be a 2 RDF for the Circulant graph $C(n; \{1,3\})$. By Lemma 2.2 and 2.3, we have
 $rd(V(G)) = \sum_{v \in V(G)} rd(v) \geq \frac{n}{5} = \frac{5m + \alpha}{5} = m + \frac{\alpha}{5}$. By Lemma 2.4, we have

$$\begin{aligned} w(f) &= 2 \times \frac{n + rd(V(G))}{6} \\ &\geq 2 \times \frac{5m + \alpha + m + \frac{\alpha}{5}}{6} \\ &= 2 \times \frac{6m + \frac{6\alpha}{5}}{6} \\ &= 2m + \left\lceil \frac{2\alpha}{5} \right\rceil. \end{aligned}$$

There are 5 cases.

Case 1. $\alpha = 0$, then $w(f) \geq 2m + 0 = 2m$.

Case 2. $\alpha = 1$, then $w(f) \geq 2m + \left\lceil \frac{2}{5} \right\rceil = 2m + 1$.

Case 3. $\alpha = 2$, then $w(f) \geq 2m + \left\lceil \frac{4}{5} \right\rceil = 2m + 1$.

Case 4. $\alpha = 3$, then $w(f) \geq 2m + \left\lceil \frac{6}{5} \right\rceil = 2m + 2$.

Case 5. $\alpha = 4$, then $w(f) \geq 2m + \left\lceil \frac{8}{5} \right\rceil = 2m + 2$.

Theorem 1.3 now follows from Lemmas 2.1 and 2.5.

In Figure 9, we shows 2 RDFs of $C(n; \{1,3\})$ for $9 \leq n \leq 14$.

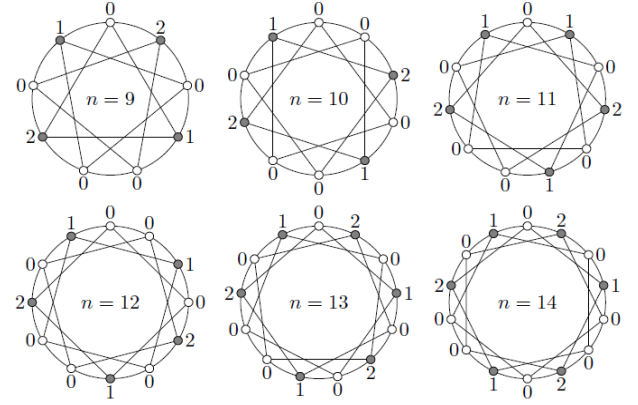


FIGURE IX. 2 RDFs of $C(n; \{1,3\})$ for $9 \leq n \leq 14$

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REFERENCES

- [1] Brešar B, Henning M A and Rall D F. Paired-domination of Cartesian products of graphs and rainbow domination. Electron. Notes Discrete Math. 22 (2005) 233-237.
- [2] Brešar B, Henning M A and Rall D F. Rainbow domination in graphs. Taiwanese J. Math. 12 (2008) 213-225.
- [3] Brešar B, Sumenjak T K. On the 2-rainbow domination in graphs, Discrete Appl. Math. 155 (2007) 2394-2400.
- [4] B. Hartnell, D.F. Rall, On dominating the Cartesian product of a graph and K_2 , Discuss. Math. Graph Theory. 24 (2004) 389-402.
- [5] B. Hartnell, D.F. Rall, Domination in Cartesian products: Vizing's Conjecture, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998, pp. 163-189.
- [6] Fu Xueliang, Yang Yuansheng, Jiang Baoqi. On the domination number of the Circulant graphs $C(n; \{1,2\})$, $C(n; \{1,3\})$ and $C(n; \{1,4\})$. Ars Combin. 102 (2011) 173-182.
- [7] Guangjun Xu. 2-rainbow domination in generalized Petersen graphs $P(n,3)$. Discrete Appl. Math. 157 (2009) 2570-2573.
- [8] Luo Meiqin. Research on Equitable Total Coloring and Rainbow Domination of Some Graphs: [dissertation]. Dalian: Dalian University of Technology, 2008.
- [9] Tong Chunling, Lin Xiaohui, Yang Yuansheng, Luo Meiqin. 2-rainbow domination of generalized Petersen graphs $P(n,2)$. Discrete Appl. Math. 157 (2009) 1932-1937.
- [10] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk. 23 (1968) 117-134.
- [11] Xueliang Fu, Yuansheng Yang, Baoqi Jiang. On the domination number of generalized Petersen graph $P(n,3)$, Ars Combin. 84 (2007) 70-81.
- [12] Yang Yuansheng, Zhao Chengye, Lin Xiaohui, Hao Xin. 3-connected 4-edge-critical non-hamiltonian graphs. J.Graph Theory. 1 (2005) 216-220.
- [13] Yunjian Wu, Nader Jafari Rad. Bounds on the 2-Rainbow Domination Number of Graphs. Graphs Combin. 29 (2013) 1125-1133.