

## The proof of strong Markov property based on one definition (II)

Tang Rong<sup>1, a</sup>, Liu Ke<sup>2, b \*</sup> and Dong yixuan<sup>3, c</sup>

<sup>1</sup>School of Economics and Management, Hainan university, P.R.China, 570228

<sup>2</sup>School of Economics and Management, Hainan university, P.R.China, 570228

<sup>3</sup>logistics management office, Hainan university, P.R.China, 570228

<sup>a</sup>tanyou01@163.com, <sup>b</sup>tangyou01@163.com, <sup>c</sup>Dongyx08@163.com

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**Abstract.** In this paper, on the basis of references [12] we continue to study strong Markov property for general Markov processes. we extend the definition of strong Markov property in [2] to generic Markov processes. and prove them are valid for every Markov processes.

### Introduction

Let  $X(t, w)$  be a measurable Markov process defining  $(\Omega, \mathcal{F}, P)$  and valuing  $(E, E)$ . To probability  $P(\Lambda(a), x_{a+t_0} \in A_0, x_{a+t_1} \in A_1)$ , when  $\Lambda(a), A_0$  are fixed,  $A_1$  varies in  $E$  to produce a measure on  $(E, E)$ . Denote as  $\bar{P}(\Lambda(a), x_{a+t_0} \in A_0, \mathcal{G})$ , that is, for each  $A_1 \in E$

$$\bar{P}(\Lambda(a), x_{a+t_0} \in A_0, A_1) = P(\Lambda(a), x_{a+t_0} \in A_0, x_{a+t_1} \in A_1)$$

In the same reason, probability  $P^{a(w_0)}(\Lambda(a(w_0)), x_{a+t_0} \in A_0, x_{a+t_1} \in A_1)$  also produce a measure on  $(E, E)$ . Denote as  $\bar{P}^{a(w_0)}(\Lambda(a(w_0)), x_{a+t_0} \in A_0, \mathcal{G})$ , that is, for each  $A_1 \in E$ , have

$$\bar{P}^{a(w_0)}(\Lambda(a(w_0)), x_{a+t_0} \in A_0, A_1) = P^{a(w_0)}(\Lambda(a(w_0)), x_{a+t_0} \in A_0, x_{a+t_1} \in A_1)$$

The above defining measure is applied to definition 1 at once. The following we generalize the definition of strong Markov property in [2] to arbitrary Markov process.

**Definition 1.** Let  $X(t, w)$  be a measurable Markov process defining on  $(\Omega, \mathcal{F}, P)$  and valuing in  $(E, E)$ .  $P(s, t; x, A)$  ( $0 \leq s < t, x \in E, A \in E$ ) be its a transition probability function. We call  $X(t, w)$  has strong Markov property, if  $X(t, w)$  satisfies:

Let  $a(w)$  be an arbitrary stopping time,  $\mathcal{F}_a @ \mathcal{S}(x_t : t \leq a)$  be the  $\mathcal{S}$ -algebra prior to  $a$  generated by  $X(t, w)$ . For each  $\Lambda(a) \in \mathcal{F}_a$ , and arbitrary finitely many  $0 \leq t_0 < t_1 < \dots < t_n; A_0, A_1, \dots, A_n \in E$ , we have

$$\begin{aligned} P(\Lambda(a), x_{a+t_v} \in A_v; 0 \leq v \leq n) &= E \left[ \int_{A_1} \int_{A_2} \mathbf{L} \int_{A_{n-2}} \int_{A_{n-1}} P(a(w) + t_{n-1}, a(w) + t_n; y_{n-1}, A_n) \right. \\ &P(a(w) + t_{n-2}, a(w) + t_{n-1}; y_{n-2}, dy_{n-1}) P(a(w) + t_{n-3}, a(w) + t_{n-2}; y_{n-3}, dy_{n-2}) \mathbf{L} \\ &\left. P(a(w) + t_1, a(w) + t_2; y_1, dy_2) \bar{P}^{a(w)}(\Lambda(a(w)), x_{a(w)+t_0} \in A_0, dy_1) \right]. \end{aligned}$$

In particular, if  $X(t, w)$  is a homogeneous Markov process, have

$$\begin{aligned} P(\Lambda(a), x_{a+t_v} \in A_v; 0 \leq v \leq n) &= \int_{A_1} \int_{A_2} \mathbf{L} \int_{A_{n-2}} \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) \\ &P(t_{n-3}, t_{n-2}; y_{n-3}, dy_{n-2}) \mathbf{L} P(t_1, t_2; y_1, dy_2) \bar{P}(\Lambda(a), x_{a+t_0} \in A_0, dy_1). \end{aligned}$$

If  $X(t, w)$  have strong Markov property,  $X(t, w)$  is called strong Markov process.

The following conclusion (1) of lemma 1 is the converse theorem of [3, theorem 4.2.1], it is the necessary condition of [3, theorem 4.2.1].

**Lemma 1.** Let  $X(t, w)$  be a measurable Markov process defining on  $(\Omega, F, P)$  and valuing in  $(E, E)$ ,  $P(s, t; x, A)$  ( $0 \leq s < t, x \in E, A \in E$ ) be a transition probability function of  $X(t, w)$ . Then, for arbitrary finitely many  $0 \leq t_0 < t_1 < \dots < t_n; A_1, \dots, A_n \in E$ , we have the following properties:

$$(1) P(x_{t_0} \in A_0, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n) =$$

$$\int_{A_0} \int_{A_1} \dots \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) \dots P(t_0, t_1; y_0, dy_1) P_{t_0}(dy_0)$$

where  $P_{t_0}(\mathbf{g})$  is a probability measure on  $(E, E)$  derived by  $x_{t_0}$ , that is,  $P_{t_0}(A) @ P(x_{t_0} \in A), A \in E$ .

$$(2) P(x_{t_1} \in A_1, \dots, x_{t_n} \in A_n | \mathcal{S}(x_{t_0}))(w) =$$

$$\int_{A_1} \dots \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) \dots P(t_0, t_1; x_{t_0}(w), dy_1), P_{\sigma(x_{t_0})}\text{-a.e..}$$

Particularly take  $n = 2, A_1 = E$ , and  $P(t_0, t_2; x_{t_0}(w), A_2)$  replace  $P(x_{t_2} \in A_2 | \mathcal{S}(x_{t_0}))(w)$  ( Because  $P(t_0, t_2; x_{t_0}(w), A_2)$  is a version of  $P(x_{t_2} \in A_2 | \mathcal{S}(x_{t_0}))(w)$  ). Then above equation is changed into K-C equation:

$$P(t_0, t_2; x_{t_0}(w), A_2) = \int_E P(t_1, t_2; y_1, A_2) P(t_0, t_1; x_{t_0}(w), dy_1), P_{\sigma(x_{t_0})}\text{-a.e..}$$

**Proof.** (1). We apply inductive method to prove this conclusion. For  $n = 1$ ,

$$\begin{aligned} P(x_{t_0} \in A_0, x_{t_1} \in A_1) &= \int_{\{x_{t_0} \in A_0\}} P(x_{t_1} \in A_1 | \mathcal{S}(x_{t_0}))(w) P(dw) = \int_{\{x_{t_0} \in A_0\}} P(t_0, t_1; x_{t_0}(w), A_1) P(dw) \\ &= \int_{A_0} P(t_0, t_1; y_0, A_1) P_{t_0}(dy_0), \text{ by the theorem of integral transformation.} \end{aligned}$$

Suppose that for  $n-1$  the conclusion holds, that is,  $P(x_{t_0} \in A_0, x_{t_1} \in A_1, \dots, x_{t_{n-1}} \in A_{n-1}) = \int_{A_0} \int_{A_1} \dots \int_{A_{n-2}} P(t_{n-2}, t_{n-1}; y_{n-2}, A_{n-1}) P(t_{n-3}, t_{n-2}; y_{n-3}, dy_{n-2}) \dots P(t_0, t_1; y_0, dy_1) P_{t_0}(dy_0)$ .

Fix  $A_0, A_1, \dots, A_{n-2}$ , but  $A_{n-1}$  vary in  $E$ . Both the left term and the right term above equality are measures on  $(E, E)$ . denote by  $\bar{P}(A_0, A_1, \dots, A_{n-2}, \mathbf{g})$  and  $\bar{m}(A_0, A_1, \dots, A_{n-2}, \mathbf{g})$  respectively. By the inductive assumption,  $\bar{P}(A_0, A_1, \dots, A_{n-2}, \mathbf{g}) = \bar{m}(A_0, A_1, \dots, A_{n-2}, \mathbf{g})$ . If  $\{x_{t_{n-1}} \in A_{n-1}\}$  is substituted into  $\bar{P}(A_0, A_1, \dots, A_{n-2}, \mathbf{g})$ , The measure on  $(\Omega, \mathcal{S}(x_{t_{n-1}}))$  is obtained. denoted by  $P(A_0, A_1, \dots, A_{n-2}, \mathbf{g})$ . So, we have  $\bar{P}(A_0, A_1, \dots, A_{n-2}, A_{n-1}) = P(A_0, A_1, \dots, A_{n-2}, \{x_{t_{n-1}} \in A_{n-1}\})$ . Hence,

$$\begin{aligned} P(x_{t_0} \in A_0, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n) &= \int_{\{x_{t_0} \in A_0, \dots, x_{t_{n-1}} \in A_{n-1}\}} P(x_{t_n} \in A_n | \mathcal{S}(x_{t_0}, \dots, x_{t_{n-1}}))(w) P(dw) \\ &= \int_{\{x_{t_0} \in A_0, \dots, x_{t_{n-1}} \in A_{n-1}\}} P(x_{t_n} \in A_n | \mathcal{S}(x_{t_{n-1}}))(w) P(dw), \text{ by Markov property,} \\ &= \int_{\{x_{t_0} \in A_0, \dots, x_{t_{n-1}} \in A_{n-1}\}} P(t_{n-1}, t_n; x_{t_{n-1}}(w), A_n) P(dw), \text{ since } P(x_{t_n} \in A_n | \mathcal{S}(x_{t_{n-1}})) = P(t_{n-1}, t_n; x_{t_{n-1}}(w), A_n), \text{ P-a.e.,} \\ &= \int_{\{x_{t_{n-1}} \in A_{n-1}\}} P(t_{n-1}, t_n; x_{t_{n-1}}(w), A_n) P(A_0, A_1, \dots, A_{n-2}, dw), \text{ by integral transformation,} \\ &= \int_{A_{n-1}} P(t_{n-1}, t_n; x_{t_{n-1}}(w), A_n) \bar{P}(A_0, A_1, \dots, A_{n-2}, dy_{n-1}), \text{ by integral transformation,} \\ &= \int_{A_{n-1}} P(t_{n-1}, t_n; x_{t_{n-1}}(w), A_n) \bar{m}(A_0, A_1, \dots, A_{n-2}, dy_{n-1}), \text{ by inductive assumption,} \\ &= \int_{A_0} \int_{A_1} \dots \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) \dots P(t_0, t_1; y_0, dy_1) P_{t_0}(dy_0). \end{aligned}$$

**Remark 1.** From third to sixth equalities also can obtain by similar to the proof of [12, (5)].

(2). By (1) and the theorem of integral transformation,

$$\begin{aligned} &\int_{\{x_{t_0} \in A_0\}} \int_{A_1} \dots \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) \dots P(t_0, t_1; y_0, dy_1) P(dw) \\ &= \int_{A_0} \int_{A_1} \dots \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) \dots P(t_0, t_1; y_0, dy_1) P_{t_0}(dy_0) \\ &= P(x_{t_0} \in A_0, x_{t_1} \in A_1, \dots, x_{t_n} \in A_n) = \int_{\{x_{t_0} \in A_0\}} P(x_{t_1} \in A_1, \dots, x_{t_n} \in A_n | \mathcal{S}(x_{t_0}))(w) P(dw) \end{aligned}$$

So by Radon-Nikodym's theorem yields (2).  $\square$

**Theorem 1.** Let  $X(t, w)$  be an arbitrary measurable Markov process defining on  $(\Omega, F, P)$  and valuing in  $(E, E)$ .  $P(s, t; x, A)$  ( $0 \leq s < t, x \in E, A \in E$ ) be one transition probability function of  $X(t, w)$ . Then  $X(t, w)$  is a strong Markov process.

**Proof.** First Let

$$L(a) = \{x_{s_1} \in B_1, L, x_{s_m} \in B_m, a \geq s_{m+1}\}, \text{ where } m \geq 1; 0 \leq s_1 \leq L \leq s_{m+1} \leq \infty; B_1, L, B_m \in E.$$

Then,

$$\begin{aligned} P(\Lambda(a), x_{a+t_v} \in A_v; 0 \leq v \leq n) &= \int_W \int_{\{L(a(w_0)), x_{a(w_0)+t_0} \in A_0\}} P(x_{a(w_0)+t_v} \in A_v; 1 \leq v \leq n | S(x_{a(w_0)+t_0})) P^{a(w_0)}(dw) P(dw_0), \text{ which follows from the} \\ &\text{same as proof of [12, theorem 1],} \\ &= \int_W \int_{\{L(a(w_0)), x_{a(w_0)+t_0} \in A_0\}} [\int_{A_1} L \int_{A_{n-1}} P(a(w_0)+t_{n-1}, a(w_0)+t_n; y_{n-1}, A_n) P(a(w_0)+t_{n-2}, a(w_0)+t_{n-1} \\ &; y_{n-2}, dy_{n-1}) L P(a(w_0)+t_0, a(w_0)+t_1; x_{a(w_0)+t_0}(w), dy_1)] P^{a(w_0)}(dw) P(dw_0), \text{ by lemma 1,} \\ &= \int_W \int_{A_1} L \int_{A_{n-1}} P(a(w_0)+t_{n-1}, a(w_0)+t_n; y_{n-1}, A_n) P(a(w_0)+t_{n-2}, a(w_0)+t_{n-1}; y_{n-2}, dy_{n-1}) L P(a(w_0)+t_1, \\ &a(w_0)+t_2; y_1, dy_2) [\int_{\{L(a(w_0)), x_{a(w_0)+t_0} \in A_0\}} P(a(w_0)+t_0, a(w_0)+t_1; x_{a(w_0)+t_0}(w), dy_1)] P^{a(w_0)}(dw) P(dw_0) \quad (1) \end{aligned}$$

,exchange the integral sequence,

$$\begin{aligned} &= \int_W [\int_{A_1} L \int_{A_{n-1}} P(a(w_0)+t_{n-1}, a(w_0)+t_n; y_{n-1}, A_n) P(a(w_0)+t_{n-2}, a(w_0)+t_{n-1}; y_{n-2}, dy_{n-1}) L \\ &P(a(w_0)+t_1, a(w_0)+t_2; y_1, dy_2) \bar{P}^{a(w_0)}(L(a(w_0)), x_{a(w_0)+t_0} \in A_0, dy_1)] P(dw_0) \quad (2) \\ &= E[\int_{A_1} L \int_{A_{n-1}} P(a(w_0)+t_{n-1}, a(w_0)+t_n; y_{n-1}, A_n) P(a(w_0)+t_{n-2}, a(w_0)+t_{n-1}; y_{n-2}, dy_{n-1}) L \\ &P(a(w_0)+t_1, a(w_0)+t_2; y_1, dy_2) \bar{P}^{a(w_0)}(L(a(w_0)), x_{a(w_0)+t_0} \in A_0, dy_1)] \end{aligned}$$

where the (2) make use of the following equality

$$\begin{aligned} &\int_{\{L(a(w_0)), x_{a(w_0)+t_0} \in A_0\}} P(a(w_0)+t_0, a(w_0)+t_1; x_{a(w_0)+t_0}(w), dy_1)] P^{a(w_0)}(dw) = \\ &P^{a(w_0)}(L(a(w_0)), x_{a(w_0)+t_0} \in A_0, x_{a(w_0)+t_1} \in A_1). \end{aligned}$$

In fact, above formula is obtained by [12, lemma 2]. In particular, if  $X(t, w)$  is a homogeneous Markov process, by the homogeneity we know (2) is changed into

$$\begin{aligned} &P(\Lambda(a), x_{a+t_v} \in A_v; 0 \leq v \leq n) \\ &= \int_W [\int_{A_1} L \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) L P(t_1, t_2; y_1, dy_2) \\ &\bar{P}^{a(w_0)}(L(a(w_0)), x_{a(w_0)+t_0} \in A_0, dy_1)] P(dw_0) \\ &= \int_{A_1} L \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) L P(t_1, t_2; y_1, dy_2) \\ &[\int_W \bar{P}^{a(w_0)}(L(a(w_0)), x_{a(w_0)+t_0} \in A_0, dy_1) P(dw_0)] \\ &= \int_{A_1} L \int_{A_{n-1}} P(t_{n-1}, t_n; y_{n-1}, A_n) P(t_{n-2}, t_{n-1}; y_{n-2}, dy_{n-1}) L P(t_1, t_2; y_1, dy_2) \bar{P}(L(a), x_{a+t_0} \in A_0, dy_1), \text{ by} \end{aligned}$$

the definition of conditional probability and [12, lemma 5].

So by  $I - p$ -system method it follows this theorem.  $\square$

**Remark 2.** For above multiple integral, we may start from inside to outside, this is the essential convention of multiple integral. also may start from outside to inside, From the proof of lemma 1 and basic idea of induction we know it is valid. In the end, we point out the associative law may be arbitrarily used to above multiple integral, that is, exchange the integral sequence is valid. We explain it by the following example:

$$\int_{A_0} \int_{A_1} \int_{A_2} \int_{A_3} P(t_3, t_4; y_3, A_4) P(t_2, t_3; y_2, dy_3) P(t_1, t_2; y_1, dy_2) P(t_0, t_1; x_{t_0}(w), dy_1) P(dw).$$

If  $A_1$  fix, and  $A_2$  vary in  $E$ ,  $P^{x_0(w)}(x_{t_1} \in A_1, x_{t_2} \in A_2)$  is changed into a measure defining on  $E$ . Denote by  $\bar{P}^{x_0(w)}(x_{t_1} \in A_1, \mathfrak{g})$ . By lemma 1 (2),

$$\int_{A_1} P(t_1, t_2; y_1, A_2) P(t_0, t_1; x_{t_0}(w), dy_1) = \bar{P}^{x_0(w)}(x_{t_1} \in A_1, A_2), \quad P_{S(x_0)}\text{-a.e.}$$

So, similar to the proof of (5) in reference [12] it follows

$$\begin{aligned} & \int_{A_0} \int_{A_1} \int_{A_2} \int_{A_3} P(t_3, t_4; y_3, A_4) P(t_2, t_3; y_2, dy_3) P(t_1, t_2; y_1, dy_2) P(t_0, t_1; x_{t_0}(w), dy_1) P(dw) \\ &= \int_{A_0} \int_{A_2} \int_{A_3} P(t_3, t_4; y_3, A_4) P(t_2, t_3; y_2, dy_3) \bar{P}^{x_0(w)}(x_{t_1} \in A_1, dy_2) P(dw) \\ &= \int_{A_0} \int_{A_2} \int_{A_3} P(t_3, t_4; y_3, A_4) P(t_2, t_3; y_2, dy_3) \left[ \int_{A_1} P(t_1, t_2; y_1, dy_2) P(t_0, t_1; x_{t_0}(w), dy_1) P(dw) \right] P(dw) \\ &= \int_{A_0} \left[ \int_{A_1} \int_{A_2} \int_{A_3} P(t_3, t_4; y_3, A_4) P(t_2, t_3; y_2, dy_3) P(t_1, t_2; y_1, dy_2) P(t_0, t_1; x_{t_0}(w), dy_1) P(dw) \right] P(dw) \end{aligned}$$

Which is the rationale of that (1) holds.

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