Solutions of a Class of Fourth Order Difference Equation with Variable Coefficient

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ABSTRACT: Difference equations is a kind of powerful tool to investigate the rule of natural phenomena, such as, physical problems arising in a wide variety of applications. In this paper, we discuss solutions of a class of fourth order difference equation with variable coefficient using the backward differences operator.

KEYWORDS: Fourth order difference equation; Variable coefficient; Solutions; Fractional difference operator

1 INTRODUCTION

It is well known that the forward difference operator is defined by

$$\Delta f(z) = f(z+1) - f(z), \quad (1)$$

and the backward difference operator is defined by

$$\nabla f(z) = f(z) - f(z-1). \quad (2)$$

Diaz & Osler (1974) [3] introduced the following positive integer order difference formulas: nth order forward difference formula

$$\Delta^n f(z) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(z+n-k), \quad (3)$$

n order backward difference formula

$$\nabla^n f(z) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(z-k), \quad (4)$$

the Leibniz rule of n order forward difference

$$\Delta^n f(z)g(z) = \sum_{k=0}^n \binom{n}{k} \Delta^{n-k} f(z) \Delta^k g(z+n-k), \quad (5)$$

and the Leibniz rule of n order backward difference

$$\nabla^n f(z)g(z) = \sum_{k=0}^n \binom{n}{k} (\nabla^{n-k} f(z-k)) (\nabla^k g(z+n-k)) \quad (6)$$

Granger & Joyeux (1980)[5], Gray & Zhang (1988)[4], and Hosking (1981)[7] discussed the following fractional difference formulas: the forward fractional difference formula

$$\Delta^\nu f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} f(x+\nu-k), \quad (7)$$

where ν is any real or complex number, the backward fractional difference formula

$$\nabla^\nu f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} f(x-k), \quad (8)$$

the Leibniz rule of the backward fractional difference

$$\nabla_x^\nu f(x)g(x) = \sum_{k=0}^n \binom{\nu}{k} \left[\nabla_x^{\nu-n} f(x-n) \right] \left[\nabla^n g(x) \right]. \quad (9)$$

Gray & Zhang (1988)[4] investigated the following properties: for α and β any complex numbers, if for α and β both are zero or positive integers, then

$$\nabla_{x_0}^\alpha \nabla_{x_0}^\beta = \nabla_{x_0}^{\alpha+\beta}, \quad (10)$$

if α is any complex number and β is not a positive integer

$$\nabla_{x_0}^\alpha \nabla_{x_0}^\beta = \nabla_{x_0}^{\alpha+\beta}. \quad (11)$$

Gray and Zhang (1988)[4] studied the solutions of the following third order difference equation

$$(ax+b)y(x) + (cx+d)y(x-1) + (ex+f)y(x-2) = H(x), \quad (12)$$

using the backward fractional difference formula. Motivated by the results given in (Atici and Elloe 2007[1]; Atici and Elloe 2009[2]; Diaz and Osler 1974[3]; Granger and Joyeux 1980[5]; Gray and Zhang 1988[4]; Holm 2011[6]; Hosking 1981[7]; Mao 1990[8]; Miller and Ross 1989[9]; Podlubny 1999[10]), in this paper, we discuss the solutions of the following fourth order difference equation

$$(a_0x+b_0)y(x) + (a_1x+b_1)y(x-1) + (a_2x+b_2)y(x-2) + (a_3x+b_3)y(x-3) = F(x). \quad (13)$$

2 RESULT

Theorem 1. Assume that there are positive numbers μ, ν such that $a_k, b_k, k = 0, 1, 2, 3$ in (13) satisfy the following relations

$$a_0 + a_3\mu^3 \neq 0, \quad (14)$$

$$a_1 = a_3\mu^2 - 2a_0\mu^{-1}, \quad (15)$$

$$a_2 = a_0\mu^{-2} - 2a_3\mu, \quad (16)$$

$$b_1 = a_3\mu^2\nu + b_3\mu^2 + a_0\mu^{-1}\nu - 2b_0\mu^{-1}, \quad (17)$$

$$b_2 = -a_3\mu\nu - 2b_3\mu - a_0\mu^{-2}\nu + b_0\mu^{-2}. \quad (18)$$

Let $y(x)$ satisfy the difference equation (13) for all x in some set A . Let

$$p_1(x) = p_{11}x + p_{12}, p_2(x) = p_{21}x + p_{22}, \quad (19)$$

where

$$p_{21} = -a_3\mu^3, p_{11} = a_0 + a_3\mu^3,$$

$$p_{22} = -a_3\mu^3\nu - b_3\mu^3, p_{12} = b_0 + a_3\mu^3\nu + b_3\mu^3. \quad (20)$$

Let $x_0 \in A$ such that $Q(x) \neq 0$ for $x \geq x_0$, where

$$Q(x) = \prod_{i=x_0}^x q(i), q(x) = \frac{p_2(x)}{p_1(x) + p_2(x)}. \quad (21)$$

Then for all $x \in A$ such that $x > x_0$ the difference equation in (13) has the equivalent form

$$\begin{aligned} \nabla_{x_0}^\nu [p_2(x) + p_1(x)]Q(x)\nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ = \mu^x F(x), \end{aligned} \quad (22)$$

Proof. For any function $\phi(x)$, let

$$\nabla = 1 - B, \quad Bf(x) = f(x-1),$$

then (see Gray and Zhang 1988)

$$\begin{aligned} & [p_2(x)\nabla + p_1(x)]\phi(x) \\ &= \{[p_2(x) + p_1(x)] - p_2(x)B\}\phi(x) \\ &= [p_2(x) + p_1(x)][\phi(x) - q(x)\phi(x-1)] \\ &= [p_2(x) + p_1(x)]Q(x)\left[\frac{\phi(x)}{Q(x)} - \frac{\phi(x-1)}{Q(x-1)}\right] \\ &= [p_2(x) + p_1(x)]Q(x)\nabla(Q(x))^{-1}\phi(x). \end{aligned} \quad (23)$$

Let $\phi(x) = \nabla_{x_0}^{-\nu+2} \mu^x y(x)$. Then

$$\begin{aligned} \nabla_{x_0}^\nu [p_2(x) + p_1(x)]Q(x)\nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ = \nabla_{x_0}^\nu [p_2(x)\nabla + p_1(x)]\nabla_{x_0}^{-\nu+2} \mu^x y(x) \end{aligned}$$

$$\begin{aligned} &= \nabla_{x_0}^\nu p_2(x)\nabla_{x_0}^{-\nu+2} \mu^x y(x) + \nabla_{x_0}^\nu p_1(x)\nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ &= \nabla_{x_0}^\nu p_2(x)\nabla_{x_0}^{-\nu+3} \mu^x y(x) + \nabla_{x_0}^\nu p_1(x)\nabla_{x_0}^{-\nu+2} \mu^x y(x). \end{aligned} \quad (24)$$

Let

$$g(x) = p_2(x), f(x) = \nabla_{x_0}^{-\nu+3} \mu^x y(x). \quad (25)$$

By the Leibniz rule of the backward fractional difference (9), we have

$$\begin{aligned} & \nabla_{x_0}^\nu p_2(x)\nabla_{x_0}^{-\nu+3} \mu^x y(x) \\ &= p_2(x)\nabla_{x_0}^\nu \nabla_{x_0}^{-\nu+3} \mu^x y(x) \\ &+ \nu \nabla p_2(x)\nabla_{x_0}^{\nu-1} \nabla_{x_0}^{-\nu+3} \mu^{x-1} y(x-1) \\ &= p_2(x)\nabla_{x_0}^3 \mu^x y(x) + \nu \nabla p_2(x)\nabla_{x_0}^2 \mu^{x-1} y(x-1) \\ &= p_2(x)\nabla^3 \mu^x y(x) + \nu \nabla p_2(x)\nabla^2 \mu^{x-1} y(x-1) \\ &= p_2(x)[\mu^x y(x) - 3\mu^{x-1} y(x-1) \\ &+ 3\mu^{x-2} y(x-2) - \mu^{x-3} y(x-3)] \\ &+ \nu p_{21}[\mu^{x-1} y(x-1) - 2\mu^{x-2} y(x-2) + \mu^{x-3} y(x-3)] \\ &= p_2(x)\mu^x y(x) + [\nu p_{21} - 3p_2(x)]\mu^{x-1} y(x-1) \\ &+ [3p_2(x) - 2\nu p_{21}]\mu^{x-2} y(x-2) \\ &+ [\nu p_{21} - p_2(x)]\mu^{x-3} y(x-3). \end{aligned} \quad (26)$$

Similarly, let

$$g(x) = p_1(x), f(x) = \nabla_{x_0}^{-\nu+2} \mu^x y(x). \quad (27)$$

By the Leibniz rule of the backward fractional difference (9), we have

$$\begin{aligned} & \nabla_{x_0}^\nu p_1(x)\nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ &= p_1(x)\nabla_{x_0}^\nu \nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ &+ \nu \nabla p_1(x)\nabla_{x_0}^{\nu-1} \nabla_{x_0}^{-\nu+2} \mu^{x-1} y(x-1) \\ &= p_1(x)\nabla^2 \mu^x y(x) + \nu \nabla p_1(x)\nabla \mu^{x-1} y(x-1) \\ &= p_1(x)[\mu^x y(x) - 2\mu^{x-1} y(x-1) + \mu^{x-2} y(x-2)] \\ &+ \nu p_{11}[\mu^{x-1} y(x-1) - \mu^{x-2} y(x-2)] \\ &= p_1(x)\mu^x y(x) + [\nu p_{11} - 2p_1(x)]\mu^{x-1} y(x-1) \end{aligned}$$

$$+[p_1(x) - \nu p_{11}] \mu^{x-2} y(x-2). \quad (28)$$

Substituting (26) and (28) in (24) and collecting terms gives

$$\begin{aligned} & \nabla_{x_0}^{\nu} [p_2(x) + p_1(x)] Q(x) \nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ &= [p_2(x) + p_1(x)] \mu^x y(x) \\ &+ [\nu p_{21} - 3p_2(x) \nu p_{11} - 2p_1(x)] \mu^{x-1} y(x-1) \\ &+ [3p_2(x) - 2\nu p_{21} + p_1(x) - \nu p_{11}] \mu^{x-2} y(x-2) \\ &+ [\nu p_{21} - p_2(x)] \mu^{x-3} y(x-3) \\ &= \mu^x \{ [p_2(x) + p_1(x)] y(x) \\ &+ [\nu p_{21} - 3p_2(x) \nu p_{11} - 2p_1(x)] \mu^{-1} y(x-1) \\ &+ [3p_2(x) - 2\nu p_{21} + p_1(x) - \nu p_{11}] \mu^{-2} y(x-2) \\ &+ [\nu p_{21} - p_2(x)] \mu^{-3} y(x-3) \} \\ &= \mu^x \{ [(p_{11} + p_{21})x + (p_{12} + p_{22})] y(x) \\ &+ [-(2p_{11} + 3p_{21})x \\ &+ (\nu p_{21} + \nu p_{11} - 2\nu p_{12} - 3\nu p_{22})] \mu^{-1} y(x-1) \\ &+ [(p_{11} + 3p_{21})x \\ &+ (-2\nu p_{21} - \nu p_{11} + p_{12} + 3p_{22})] \mu^{-2} y(x-2) \\ &+ [-p_{21}x + \nu p_{21} - p_{22}] \mu^{-3} y(x-3) \}. \end{aligned} \quad (29)$$

Using (15) to (19), from (29) we have

$$\begin{aligned} & \nabla_{x_0}^{\nu} [p_2(x) + p_1(x)] Q(x) \nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ &= \mu^x [(a_0x + b_0)y(x) + (a_1x + b_1)y(x-1) \\ &+ (a_2x + b_2)y(x-2) + (a_3x + b_3)y(x-3)]. \end{aligned} \quad (30)$$

Thus, the difference equation (13) has the equivalent form (22), since the above process can be reversed.

Corollary 1. For the fourth order difference equation

$$\begin{aligned} & (a_0x + b_0)y(x) + (a_1x + b_1)y(x-1) \\ &+ (a_2x + b_2)y(x-2) + (a_3x + b_3)y(x-3) = 0. \end{aligned} \quad (31)$$

Assume that there are positive numbers μ, ν such that a_k, b_k, k in (31) satisfy the relations (15) to (18). Then

$$y(x) = C \mu^{-x} \nabla_{x_0}^{\nu-2} Q(x) \quad (32)$$

is a solution of (31), where C is an arbitrary constant, $Q(x)$ is defined by (21).

Proof. By the Theorem 1, we see that the difference equation (31) has the equivalent form

$$\begin{aligned} & \nabla_{x_0}^{\nu} [p_2(x) + p_1(x)] Q(x) \nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x y(x) \\ &= 0. \end{aligned} \quad (33)$$

From (9), we have

$$\begin{aligned} & \nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x C \mu^{-x} \nabla_{x_0}^{\nu-2} Q(x) \\ &= \nabla C(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \nabla_{x_0}^{\nu-2} Q(x) \\ &= \nabla C(Q(x))^{-1} Q(x) \\ &= \nabla C \\ &= 0. \end{aligned} \quad (34)$$

Thus

$$\begin{aligned} & \nabla_{x_0}^{\nu} [p_2(x) + p_1(x)] Q(x) \nabla(Q(x))^{-1} \nabla_{x_0}^{-\nu+2} \mu^x [C \mu^{-x} \nabla_{x_0}^{\nu-2} Q(x)] \\ &= 0. \end{aligned} \quad (35)$$

(32) is the solution of (33), thus (32) is the solution of (31).

3 EXAMPLE

In this section, we consider the following difference equation

$$\begin{aligned} & (3x+2)y(x) - (14x+4)y(x-1) \\ &+ (20x+8)y(x-2) - 8(x+2)y(x-3) = 0. \end{aligned} \quad (36)$$

Let $\nu = 2, \mu = 1/2$. We have

$$\begin{aligned} & a_0 + a_3 \mu^3 = 3 - 8/8 = 2 \neq 0, \\ & a_1 = -14 = -8/4 - 2 \cdot 3 \cdot 2 = a_3 \mu^2 - 2a_0 \mu^{-1}, \\ & a_2 = 20 = 3 \cdot 4 - 2 \cdot (-8)/2 = a_0 \mu^{-2} - 2a_3 \mu, \\ & b_1 = -4 = -8/2 - 16/4 + 3 \cdot 2 \cdot 2 - 2 \cdot 2 \cdot 2 \\ &= a_3 \mu^2 \nu + b_3 \mu^2 + a_0 \mu^{-1} \nu - 2b_0 \mu^{-1}, \\ & b_2 = 8 = 8 + 2 \cdot 16/2 - 3 \cdot 4 \cdot 2 + 2 \cdot 4 \\ &= -a_3 \mu \nu - 2b_3 \mu - a_0 \mu^{-2} \nu + b_0 \mu^{-2}, \\ & a_k, b_k, k = 0, 1, 2, 3 \text{ in (36) satisfy the relations (15) to (18) in Theorem 1. Using (20), we get} \end{aligned}$$

$$p_{21} = 1, p_{11} = 2, p_{22} = 4, p_{12} = -2. \quad (37)$$

Using (19) and (21), we get

$$p_1(x) = 2x - 2, p_2(x) = x + 4, q(x) = \frac{x+4}{3x+2}. \quad (38)$$

By Corollary 1, we see that

$$y(x) = C 2^x \prod_{i=-3}^x q(i). \quad (39)$$

is a solution of (36), where C is an arbitrary constant.

Next, we check that (39) is a solution of (36). From (39), we have

$$\begin{aligned} y(x) &= C2^x Q(x) = C2^x \prod_{i=-3}^x q(i) \\ &= C2^x \frac{x+2}{3x-4} \frac{x+3}{3x-1} \frac{x+4}{3x+2} \prod_{i=-3}^{x-3} q(i). \end{aligned} \quad (40)$$

$$\begin{aligned} y(x-1) &= C2^{x-1} Q(x-1) = C2^{x-1} \prod_{i=-3}^{x-1} q(i) \\ &= C2^{x-1} \frac{x+2}{3x-4} \frac{x+3}{3x-1} \prod_{i=-3}^{x-3} q(i). \end{aligned} \quad (41)$$

$$\begin{aligned} y(x-2) &= C2^{x-2} Q(x-2) = C2^{x-2} \prod_{i=-3}^{x-2} q(i) \\ &= C2^{x-2} \frac{x+2}{3x-4} \prod_{i=-3}^{x-3} q(i). \end{aligned} \quad (42)$$

$$y(x-3) = C2^{x-3} Q(x-3) = C2^{x-3} \prod_{i=-3}^{x-3} q(i). \quad (43)$$

Substituting (40), (41), (42) and (43) in (36) and collecting terms gives

$$\begin{aligned} &= C2^{x-3} \prod_{i=-3}^{x-3} q(i) \left[8(3x+2) \frac{x+2}{3x-4} \frac{x+3}{3x-1} \frac{x+4}{3x+2} \right. \\ &\quad - 4(14x+4) \frac{x+2}{3x-4} \frac{x+3}{3x-1} \\ &\quad \left. + 2(20x+8) \frac{x+2}{3x-4} - 8(x+2) \right] \\ &= C2^{x-3} \prod_{i=-3}^{x-3} q(i) \left[\frac{8(x+2)(x+3)(x+4)}{(3x-4)(3x-1)} \right. \\ &\quad - \frac{4(x+2)(x+3)(14x+4)}{(3x-4)(3x-1)} + \\ &\quad + \frac{2(x+2)(20x+8)(3x-1)}{(3x-4)(3x-1)} \\ &\quad \left. - \frac{8(x+2)(3x-4)(3x-1)}{(3x-4)(3x-1)} \right] \\ &= C2^{x-3} \prod_{i=-3}^{x-3} q(i) 4(x+2) \left[\frac{2(x+3)(x+4)}{(3x-4)(3x-1)} \right. \\ &\quad - \frac{(x+3)(14x+4)}{(3x-4)(3x-1)} + \frac{(10x+4)(3x-1)}{(3x-4)(3x-1)} \\ &\quad \left. - \frac{2(3x-4)(3x-1)}{(3x-4)(3x-1)} \right] = 0. \end{aligned} \quad (44)$$

Therefore (39) is a solution of (36).

4 CONCLUSIONS

Difference equations is a kind of powerful tool to investigate the rule of natural phenomena, such as, physical problems arising in a wide variety of applications. In this paper, we discuss solutions of a class of fourth order difference equation with variable coefficient by the backward differences operator. The fourth order difference equation

$$(a_0x + b_0)y(x) + (a_1x + b_1)y(x-1) + (a_2x + b_2)y(x-2) + (a_3x + b_3)y(x-3) = F(x)$$

has the equivalent form

$$\begin{aligned} &\nabla_{x_0}^v [p_2(x) + p_1(x)] Q(x) \nabla(Q(x))^{-1} \nabla_{x_0}^{-v+2} \mu^x y(x) \\ &= \mu^x F(x). \end{aligned}$$

ACKNOWLEDGEMENTS

This research was supported by Guangxi Natural Science Foundation (Project No. 2012GXNS-FAA053009) and Scientific Research Foundation of the Education Department of Guangxi Autonomous Region of China (No. 201010LX473, KY2015ZD103, KY2015LX340, Key Projection of Hechi University of China (No. 2014ZD03). The corresponding author is Yun Ouyang.

REFERENCES

- [1] Atici F. M. & Eloe P. W. 2007. A transform method in discrete fractional calculus. *International Journal of Difference Equations* 2 (2): 165-176.
- [2] Atici F. M. & Eloe P. W. 2009. Initial value problems in discrete fractional calculus. *Proceedings of the American Mathematical Society* 137 (3): 981-989.
- [3] Diaz J. B. & Osler T. J. 1974. Differences of fractional order. *Mathematics of Computation* 28 (125): 185-201.
- [4] Gray H. L. & Zhang N. F. 1988. On a new definition of the fractional difference, *Mathematics of Computation* 50 (182): 513-529.
- [5] Granger C. W. J. & Joyeux R. 1980. An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis* 1 (1): 15-29.
- [6] Holm M. T. 2011. *The Theory of discrete fractional calculus: development and application, Dissertations, Theses, and Student Research Papers in Mathematics*, Nebraska-Lincoln: University of Nebraska-Lincoln.
- [7] Hosking J. R. M. 1981. Fractional differencing, *Biometrika* 68 (1): 165-176.
- [8] Mao J. Z. 1990. *Basic combinatorial mathematics*, Wuhan: China Central Normal University publishing house.
- [9] Miller K. S. & Ross B. 1989. *Fractional difference calculus. In Univalent functions, fractional calculus, and their applications (Koriyama, 1988), Ellis Horwood Ser. Math. Appl., pages 139-152*. Horwood: Chichester.
- [10] Podlubny I. 1999. *Fractional Differential Equations*, New York: Academic Press.