A simple time domain collocation method for solving the van der Pol oscillator

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Keywords

Time domain collocation method; high dimensional harmonic balance method; period solution; van der Pol equation; limit cycle.

Abstract

In this paper, the time domain collocation method (TDC) is applied to solve the van der Pol equation. The results obtained by the TDC are compared with the results by the traditional harmonic balance method (HB) and high dimensional harmonic balance method (HDHB). For the extended TDC method, appropriately increasing the number of collocation points can significantly relive the nonphysical solution phenomenon and generate a better solution. Compared with the HDHB, the TDC method is derived more strictly, and more convenient to implement. Numerical results verify the simplicity, accuracy and efficiency of the present TDC method.

Introduction

For nonlinear problems, close form exact solution is rarely available. During the research of nonlinear phenomena, various approximate and numerical methods were proposed. Among the analytical methods, the perturbation method [1] has been dominating in solving nonlinear problems. But it still has some limitations. The harmonic balance method (HB) and high dimensional harmonic balance method (HDHB) [2-3] have achieved some success as alternative to time marching integration methods. However, the well known HB method suffers from its tedious symbolic operations required by processing the nonlinear term in system, and the HDHB method was hurt by its generating additional meaningless fake solutions.

Dai, Schnoor and Atluri (2012) [4] have recently proposed a time domain collocation method, and applied it to solve the Duffing oscillator [Dai, Yue and Yuan (2013)] [5]. Compared with the other methods that have popularly used, such as the high dimensional harmonic balance method (HDHB) and the harmonic balance method (HB), it is shown that the TDC method is simpler and easier to implement. The TDC method avoids the symbolic calculations and also restrains the generation of fake solutions.

Herein we focus on solving the van der Pol oscillator by using the TDC method. The TDC solutions as well as the HDHB and HB solutions are derived for both the unforced and forced van der Pol oscillator. The structure of this paper is organized as follows. In Section 2, the time domain collocation method (TDC) as well as the extended TDC method is formulated to find periodic solutions. In Section 3, the results of TDC, HDHB and HB are displayed, and the comparisons between them are made for unforced and forced cases separately. Finally, we come to some conclusions in Section 4.

The methodology

The time domain collocation method

The van der Pol oscillator generally takes the form

$$\ddot{x} - \alpha(1-x^2)\dot{x} + x = F\cos\omega_1 t\,. \tag{1}$$

For the case with no external force, i.e. $F=0$, one obtains free oscillations. For the forced oscillator, i.e. $F\neq 0$, forced oscillations will occur.

Firstly, we start with an unforced van der pol oscillator. In this case, the periodic solution may be expressed as

$$x(t) = A_0 + \sum_{n=1}^{N} A_n\cos n\omega t + B_n\sin n\omega t\,, \tag{2}$$

which only contains the natural frequency ω. Here N is the number of harmonics used in the approximation, and A_0, A_n, B_n ($n=1,\ 2,\ \ldots$, N) are the TDC solution coefficient variables.

Then we can obtain the residual error function by substituting the approximate solution, the expression (2), into the following equation

$$R(t) = \ddot{x} - \alpha(1-x^2)\dot{x} + x \neq 0\,. \tag{3}$$

To obtain the periodic solution of the van der Pol oscillator, we enforce $R(t)$ to be zero at 2N+1 points, which are equally spaced in a period of the oscillator. The points are chosen as

$$t_i = 2(i-1)\pi/((2N+1)\omega)\ (i=1,\ \ldots,\ 2N+1)\,. \tag{4}$$

Thus we obtain a system of 2N+1 nonlinear algebraic equations:

$$R_i = \ddot{x}(t_i) - \alpha(1-x^2(t_i))\dot{x}(t_i) + x(t_i) = 0_i\,, \tag{5}$$

which is named the TDC algebraic system for the periodic solution.

According to the expression (2), we have

$$x(t_i) = A_0 + \sum_{n=1}^{N} A_n\cos n\omega t_i + B_n\sin n\omega t_i\,, \tag{6a}$$

$$\dot{x}(t_i) = \omega\sum_{n=1}^{N} -nA_n\sin n\omega t_i + nB_n\cos n\omega t_i\,, \tag{6b}$$

$$\ddot{x}(t_i) = -\omega^2\sum_{n=1}^{N} n^2A_n\cos n\omega t_i + n^2B_n\sin n\omega t_i\,. \tag{6c}$$

Thus we can derive the Jacobian matrix $\mathbf{J}$ to the algebraic system

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial R_i}{\partial A_j} & \dfrac{\partial R_i}{\partial B_j} \end{bmatrix}_{(2N+1)\times(2N+1)}, \tag{7}$$

with

$$\frac{\partial R_i}{\partial A_j} = -\omega^2 j^2 \cos j\omega t_i + \alpha\omega j \sin j\omega t_i + 2\alpha x(t_i)\dot{x}(t_i)\cos j\omega t_i - \alpha x^2(t_i)\omega j \sin j\omega t_i + \cos j\omega t_i ,$$

$$\frac{\partial R_i}{\partial B_j} = -\omega^2 j^2 \sin j\omega t_i - \alpha\omega j \cos j\omega t_i + 2\alpha x(t_i)\dot{x}(t_i)\sin j\omega t_i + \alpha x^2(t_i)\omega j \cos j\omega t_i + \sin j\omega t_i .$$

Thereby, the coefficient variables A_n, B_n can be determined by the Newton-Raphson method.

For the forced van der Pol oscillator, the motion is more complex. Both the fundamental frequency ω and the external forcing frequency ω_1 contribute to the oscillation. Thus these two general incommensurate reference frequencies are included in the approximate solution. So the following truncated Fourier series is used.

$$x(t) = \sum_{n=0}^{N}\sum_{m=-M}^{M}(x_{n,m}\cos(n\omega + m\omega_1)t + y_{n,m}\sin(n\omega + m\omega_1)t) , \quad (8)$$

Similar to the unforced system, by collecting 2(N+1)(2M+1) points in a period T of forced van der Pol oscillator, we get a system of 2(N+1)(2M+1) nonlinear algebraic equations.

$$R_i = \ddot{x}(t_i) - \alpha(1 - x^2(t_i))\dot{x}(t_i) + x(t_i) - F\cos\omega_1 t_i = 0_i . \quad (9)$$

Considering the expression (8), we have

$$x(t_i) = \sum_{n=0}^{N}\sum_{m=-M}^{M} x_{n,m}\cos(n\omega + m\omega_1)t_i + y_{n,m}\sin(n\omega + m\omega_1)t_i , \quad (10a)$$

$$\dot{x}(t_i) = \sum_{n=0}^{N}\sum_{m=-M}^{M} -(n\omega + m\omega_1)x_{n,m}\sin(n\omega + m\omega_1)t_i + (n\omega + m\omega_1)y_{n,m}\cos(n\omega + m\omega_1)t_i \quad (10b)$$

$$\ddot{x}(t_i) = \sum_{n=0}^{N}\sum_{m=-M}^{M} -(n\omega + m\omega_1)^2 x_{n,m}\cos(n\omega + m\omega_1)t_i - (n\omega + m\omega_1)^2 y_{n,m}\sin(n\omega + m\omega_1)t_i \quad (10c)$$

Thus the Jacobian matrix $\mathbf{J}$ of system (9) can be obtained.

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial R_i}{\partial x_{n,m}} & \dfrac{\partial R_i}{\partial y_{n,m}} \end{bmatrix}_{(2(N+1)(2M+1))\times(2(N+1)(2M+1))} , \quad (11)$$

with

$$\frac{\partial R_i}{\partial x_{n,m}} = -(n\omega + m\omega_1)^2\cos(n\omega + m\omega_1)t_i + \alpha(1 - x^2(t_i))(n\omega + m\omega_1)\sin(n\omega + m\omega_1)t_i + 2\alpha x(t_i)\dot{x}(t_i)\cos(n\omega + m\omega_1)t_i + \cos(n\omega + m\omega_1)t_i ,$$

$$\frac{\partial R_i}{\partial y_{n,m}} = -(n\omega + m\omega_1)^2\sin(n\omega + m\omega_1)t_i - \alpha(1 - x^2(t_i))(n\omega + m\omega_1)\cos(n\omega + m\omega_1)t_i + 2\alpha x(t_i)\dot{x}(t_i)\sin(n\omega + m\omega_1)t_i + \sin(n\omega + m\omega_1)t_i .$$

Then the Newton-Raphson method or the novel globally optimal iterative algorithm (GOIA) can help us determine the coefficients $x_{n,m}$ and $y_{n,m}$. The GOIA method is based on the concept of best decent vector u which takes the form of $u = \alpha_c F + B^T F$. It is proposed to solve a system of nonlinear algebraic equations (NAEs) without inverting the Jacobian matrix at each step. For more details of GOIA, refer to Liu and Atluri (2012) [3].

The extended time domain collocation method

In the previous subsection, we have shown how to implement the TDC method. To obtain a good prediction, the number N of harmonics in the trial solution should be big enough. When N is small, the results may not be up to our expectation. To improve this, we can use more collocation points K than 2N, where N is the number of harmonics included in the TDC analysis. For example, when solving the unforced van der Pol oscillator, K points are used to build a TDC resulting system of K equations, $K > 2N$.

Since the number of equations outnumbered the number of unknowns, we seek to minimize the function $f = \sum_{i=1}^{K} R_i^2$. Thus we require

$$\frac{\partial f}{\partial x_j} = 2\sum_{i=1}^{K} R_i \frac{\partial R_i}{\partial x_j} = 0 . \tag{12}$$

It can be written in a vector form:

$$\mathbf{Fx} = \mathbf{J}^T \mathbf{R} = \mathbf{0} , \tag{13}$$

where

$\mathbf{R} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ \mathrm{R}_k \end{bmatrix}$, and $\mathbf{J}$ be the Jacobian matrix of $\mathbf{R}$.

To solve Eq. (13), we need to find the explicit expression for the Jacobian matrix $\mathbf{B}$ of the system. It can be easily obtained in a way like this:

$$\mathbf{B} = \begin{bmatrix} \dfrac{\partial \mathbf{Fx}}{\partial x_i} & \dfrac{\partial \mathbf{Fx}}{\partial y_i} \end{bmatrix}. \tag{14}$$

Here

$$\frac{\partial \mathbf{Fx}}{\partial x_i} = \mathbf{J}^T \frac{\partial \mathbf{R}}{\partial x_i} + \frac{\partial \mathbf{J}^T}{\partial x_i}\mathbf{R}, \quad \frac{\partial \mathbf{Fx}}{\partial y_i} = \mathbf{J}^T \frac{\partial \mathbf{R}}{\partial y_i} + \frac{\partial \mathbf{J}^T}{\partial y_i}\mathbf{R} .$$

Thus

$$\mathbf{B} = \begin{bmatrix} \mathbf{J}^T \dfrac{\partial \mathbf{R}}{\partial x_i} + \dfrac{\partial \mathbf{J}^T}{\partial x_i}\mathbf{R} & \mathbf{J}^T \dfrac{\partial \mathbf{R}}{\partial y_i} + \dfrac{\partial \mathbf{J}^T}{\partial y_i}\mathbf{R} \end{bmatrix} = \mathbf{J}^T\mathbf{J} + \begin{bmatrix} \dfrac{\partial \mathbf{J}^T}{\partial x_i} & \dfrac{\partial \mathbf{J}^T}{\partial y_i} \end{bmatrix}\mathbf{R} . \tag{15}$$

As R_i converges to zero during the iteration, the second term at the right hand of Eq. (15) becomes useless. So we neglect it and Eq. (15) reduces to

$$\mathbf{B} = \mathbf{J}^T \mathbf{J} \,. \tag{16}$$

In subsection 2.1, we have obtained $\mathbf{R}$ and $\mathbf{J}$ for the original TDC method, thus $\mathbf{Fx}$ and $\mathbf{B}$ can be obtained conveniently. Then we can use the Newton-Raphson or GOIA method to solve the NAEs.

Results and discussion

The unforced oscillator

Using a classic numerical integration scheme known as Runge-Kutta method, we get the time marching solution of the oscillator which serves as the benchmark solution. Figure 1 shows the frequency results from the HB method with various numbers of harmonics, in comparison with the time marching solution for α up to 5.0.

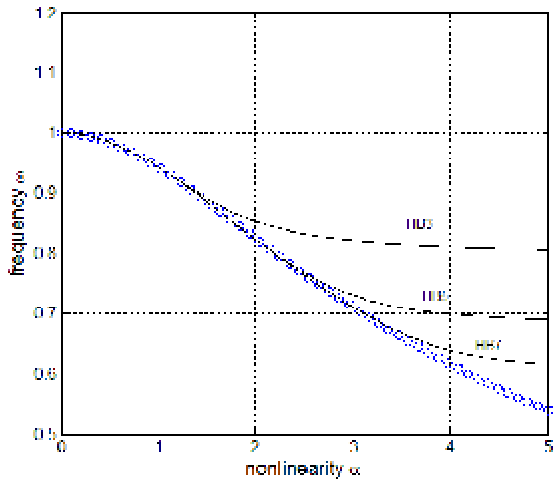


Fig. 1. Comparisons of the fundamental frequency ω versus the non-linearity coefficient α for the unforced oscillator. Solid line: HB solutions with various numbers of harmonics; open circle: time marching results.

The trouble in implementing HB method is to find the analytical expressions for the nonlinear functions r_i and s_i. In this study, we used Mathematica to calculate them.

Figure 1 shows that as the number of harmonics increases, the results from the HB method do have an excellent agreement with the time marching solution.

In Section 2, The NAEs of different methods for solving van der Pol oscillator have been given. To solve them, the initial values for Newton iterative process have to be selected carefully, because improper initial values will direct the algebraic system to undesired solutions. Thus in the following discussions, the HB solutions are used to provide the initial values for TDC and HDHB method.

The HDHB3 and HDHB4 solutions are displayed in Fig. 2 denoted by plus and filled dot separately. We can see that HDHB4 curve follows the time marching solution a little far beyond the HDHB3 solution, thus the even harmonics are not sufficiently small to be neglected. The higher order HDHB solutions will also prove that.

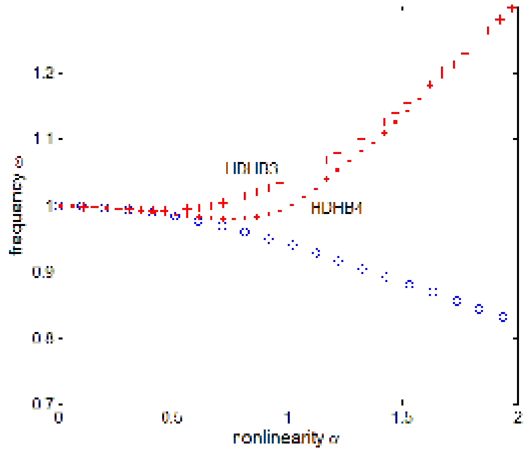
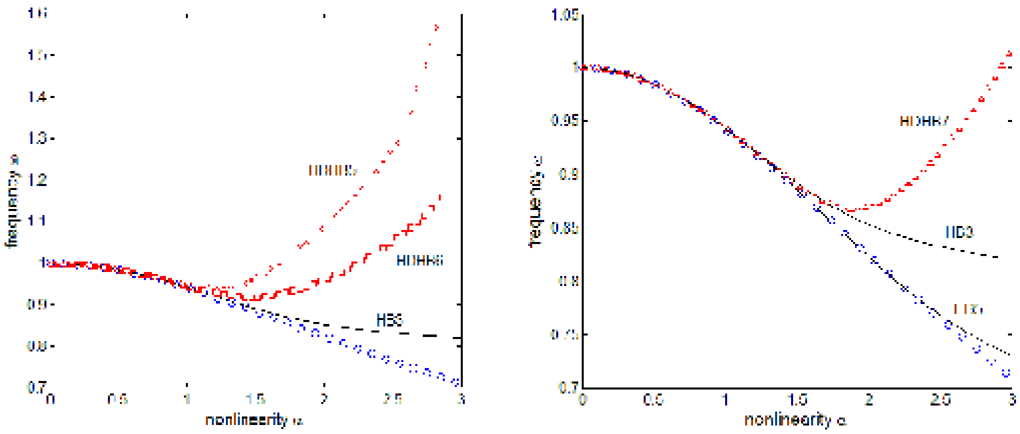


Fig. 2. For the unforced oscillator, the frequency results from the HDHB method, in comparison with the time marching results: HDHB results with three harmonics (plus) and four harmonics (filled dot); open circle: time marching results.

To get a better approximation by HDHB method, we increase the number of harmonics. The results are shown in figure 3.



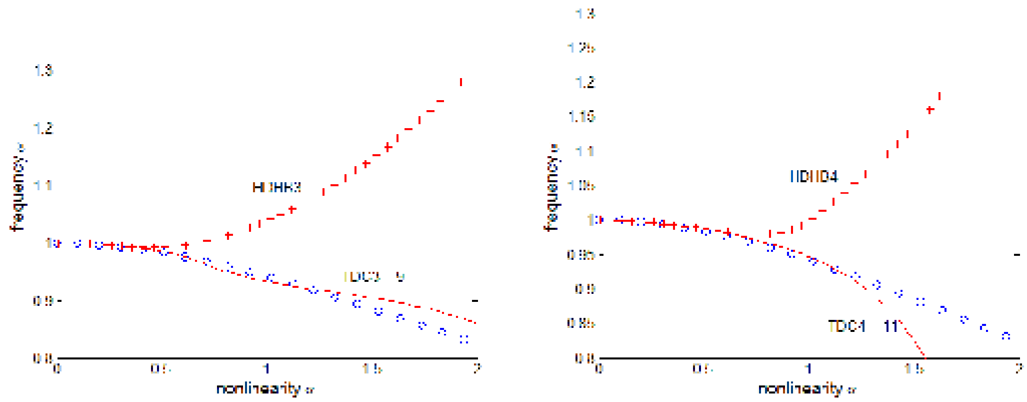
(a) (b)

Fig. 3. For the unforced oscillator, the frequency results from the HDHB method, in comparison with the results from the time marching and the HB methods: (a) HDHB results with five harmonics (cross) and six harmonics (square); (b) HDHB results with seven harmonics (triangle). Solid line: HB solutions; open circle: time marching results.

As we can see in Fig. 3 (a), the frequency response curve from HDHB6 follows the time marching solution for a longer distance than the HDHB5. Thus the statement above is confirmed.

In Fig. 3 (b), we find the HDHB7 solution follows the HB3 solution even after the HB3 curve deviates from the time marching results. According to the article of Liu, Dowell and Hall (2006) [2], it is concluded that the HDHB2n+1 results have the same accuracy as the HBn results. However, this statement is not absolutely right. Actually, as the number of harmonics used in the HB analysis increases, it becomes harder and harder for the results from HDHB2n+1 to follow the HBn results.

Then we increase the number of collocation points in TDC analysis, thus we can obtain the results of the extended TDC method.



(a) (b)

Fig. 4. For the unforced oscillator, the results from the extended TDC method, in comparison with the results from the time marching and the HDHB results: (a) three harmonics, nine collocation points; (b) four harmonics, eleven collocation points. Solid line: TDC results; plus: HDHB results; open circle: time marching results.

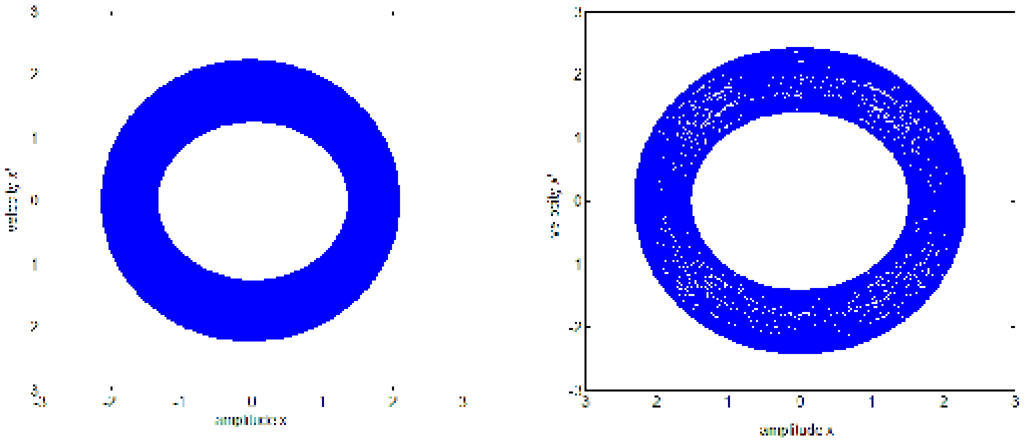
In this study, TDCn_m refers to the TDC method with n harmonics and m collocation points. The results from TDC3_9 and TDC4_11, which are denoted by solid lines, are displayed in Fig. 4 (a) and (b) respectively. Compared with the HDHB solutions (denoted by the plus), the extended TDC curve can follow the time marching results for a larger value of α. In Fig. 4 (a), the TDC3_9 curve matches well with the time marching results until $\alpha > 0.6$ As to the TDC4_11 curve, we can see that in Fig. 4 (b), it keeps very close to time marching results even when $\alpha = 1.0$. Therefore, the extended TDC method yields better results than the HDHB method.

However, to the higher order results of TDC method, the increase of the number of collocation points does not improve the accuracy obviously, because the number of harmonics itself is large enough to provide a sufficiently accurate solution. When the number of collocations approaches infinity, the least square method is obtained.

The results demonstrated above are all generated by frequency matching procedure. That is, at each step, the previous solution is employed as the initial condition for the next step. And the original initial value is provided by the HB method. Here, we provide another way for the initial value generation. Through Monte Carlo simulation, the initial values are randomly generated for a large number of computations. As the Newton or GOIA method is sensitive to initial values, the extended TDC method will provide a lot of different results with all the initial values. However, as the number of collocation points increases, the results will converge to several particular solutions.

The forced oscillator

To check the practicability of TDC method in the forced case of van der Pol oscillator, the TDC solution is also derived. The phase plane for the motions resumed from the TDC solution is plotted in Fig. 5 (a), in comparison with the HB solution, which is plotted in Fig. 5 (b).



(a) (b)

Fig. 5. For the forced oscillator, the phase planes resumed from the TDC and HB solutions

The approximate solution takes the form

$$x(t) = \sum_{n=0}^{1} \sum_{m=-1}^{1} x_{n,m} \cos(n\omega + m\omega_1)t + y_{n,m} \sin(n\omega + m\omega_1)t \,. \tag{36}$$

We can use the TDC scheme proposed in Section 2 to determine the coefficients $x_{n,m}$ and $y_{n,m}$. It needs to be noticed that the values of period T and the number of collocation points can be chosen separately. Since the motion of the forced oscillator looks like 'mild' chaos, the period of the oscillator is uncertain to us. Thus a good choice of the 'period' T and the number of collocation points K is necessary to a good prediction. After numerous numerical simulations, we do find that some combinations of T and N give good results. In Fig. 8, we choose T=33, and K=11. The harmonics ω, ω_1, and $\omega \pm \omega_1$ are included. As we can see, the TDC solution matches well with the HB solution. Except for that, some other combinations of T and N can also provide reasonable results, e.g. T=23, K=11; T=39, K=13, etc.

Conclusions

In this paper, the time domain collocation (TDC) method is applied to solve the van der Pol oscillator. By enforcing the residual error to be zero at discrete time intervals on a period of the oscillator, the resulting nonlinear algebraic equations of TDC method can be easily obtained. Then the resultant NAEs were solved by the Newton-Raphson method or GOIA method. Unlike the HDHB method, there is no need to do the Fourier transformation in the implementation of TDC method. It makes TDC method simpler than HDHB method, and reduces the computational cost.

Then the TDC solutions are compared with the results obtained from HDHB, HB and time marching methods. For the unforced van der Pol oscillator, it is demonstrated that the original TDC method gives the same results of HDHB method, while the extended TDC method remarkably improved the approximation when the number of harmonics included in the analysis is small. For the forced van der Pol oscillator, we compared the results of TDC method and HB method. We find that in the forced case, the implementation of TDC method is a little complex because of the quasi-periodic motion of the trajectory. When sufficient terms of those two frequencies are included, we can obtain some reliable results at some combinations of time interval T and number K of collocation points.

Acknowledgements

This study is financially supported by the Chinese NSF (11172235)

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