Some properties and applications of (inverse) $L_{n-1}$-matrices

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Abstract. In this paper we study some combinatorial properties and inequalities of some classes of $Z$-matrices. These matrices arise in many problems in the mathematical and physical sciences. We show that all (inverse) $L_{n-1}$-matrices are irreducible and some eigenvalue inequalities of (inverse) $L_{n-1}$-matrices.

Introduction and Notation

Throughout we deal with $n \times n$ $Z$-matrices, i.e. real matrices whose off-diagonal entries are nonpositive. These matrices arise in many problems in the mathematical and physical sciences. Some of the best-known subclasses of $Z$-matrices are the class of $M$-matrices (introduced by Ostrowski), the class of $L_{n-1}$-matrices (introduced by Ky Fan and G. A. Johnson), and the class of $F_0$ -matrices (introduced by G. A. Johnson). Especially for $M$-matrices, a large number of properties and characterizations exist. However, the other classes of $Z$-matrices are also of great interest. In this paper we study some combinatorial properties and inequalities of $L_{n-1}$-matrices and inverse $L_{n-1}$-matrices. Firstly we introduce some definitions.

In 1992 Fiedler and Markham [1] introduced the following classification of $Z$-matrices:

Definition 1 Let $L_s$ (for $s=0,1,\ldots, n$) denote the class of matrices consisting of real $n \times n$ matrices which have the form

\[ A = tI - B, \text{ where } B \geq 0 \text{ and } \rho_s(B) \leq t < \rho_{s+1}(B), \]

here

\[ \rho_s(B) = \max \{ \rho(B) : B \text{ is an } s \times s \text{ principal submatrix of } B \}, \]

and we set $\rho_0(B) = -\infty$ and $\rho_{n+1}(B) = \infty$.

The scalar $t$ and the matrix $B$ in (1) are not unique, but every $Z$-matrix belongs to exactly one set $L_s$. Moreover, none of the class $L_s$ is void. However, if one considers a fixed matrix $B$ in (1), some of the class $L_s$ can be the same, since we have in general only that

\[ \rho_1(B) \leq \rho_2(B) \leq \cdots \leq \rho_n(B) \]

The class $L_s$ is just the class of $n \times n$ (singular and nonsingular) $M$-matrices. The class $L_{n-1}$ is introduced by G. A. Johnson [2], and this class contains the $N$-matrices defined by Ky Fan [3]. Moreover, the class of $n \times n$ $F_0$ -matrices introduced by Johnson [2] is just $L_{n-2}$. Here we should mention that the classification of $Z$-matrices given above inherits the dimension of the matrices one considers. If we deal with $n \times n$ matrices, we have $n+1$classes of $Z$-matrices, each consisting of matrices of the same dimension.

As proved in [1], for each $s$ with $1 \leq s \leq n - 1$, the class $L_s$ is equal to the class of $Z$-matrices for which all principal submatrices of order $s$ are $M$-matrices, but there exists a principal submatrix of order $s+1$ which is not an $M$-matrix. Additional properties of some of these classes are given in [4, 5, 6].

On the other hand, there has been interest in inverse $M$-matrices, i.e., any nonsingular matrix $B \geq 0$ whose inverse is an $M$-matrix. A survey of this topic is given by C. R. Johnson [7]. Thus, it is natural to determine classes of matrices which are inverse $Z$-matrices. A first step was taken by G. A. Johnson [8], who proved that a matrix of negative type D is an inverse $L_{n-1}$-matrix. Later, Chen [9]
studied necessary conditions for a matrix to be an inverse $F_0$-matrix; some new results on
$Z$-matrices are in [10].

At last, for two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the Hadamard product of $A \circ B$ is
defined and denoted by $A \circ B = (a_{ij}b_{ij})$.

In this paper, we show that all (inverse) $L_{n-1}$-matrices are irreducible and some eigenvalue
inequalities of (inverse) $L_{n-1}$-matrices.

**Result and proof**

In this section, we firstly introduce some combinatorial properties of $L_{n-1}$-matrices and inverse $L_{n-1}$-matrices.

**Theorem 1:** All (inverse) $L_{n-1}$-matrices are irreducible.

**Proof:** We firstly show that all $L_{n-1}$-matrices are irreducible.

Assume that a $L_{n-1}$-matrix $A$ is reducible, then there exists a permutation matrix $P$ such that

$$P^TAP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} = I - \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix},$$

(2)

where $A_1$ and $A_2$ are $s \times s$ ($1 \leq s < n$) and $(n-s) \times (n-s)$ matrices respectively, $B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} \geq 0$.

Since $A \in L_{n-1}$, then $P^TAP \in L_{n-1}$ and $\rho_{n-1}(B) \leq t < \rho(B)$. But $\rho(B) = \max \{\rho(B_1), \rho(B_3)\}$, then $\rho(B) \leq \rho_{n-1}(B)$, this is a contradiction, so all $L_{n-1}$-matrices are irreducible.

At lastly, we prove that all inverse $L_{n-1}$-matrices are irreducible.

Assume that an inverse $L_{n-1}$-matrix $A$ is reducible, then there exists a permutation matrix $P$ such that

$$P^TAP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} = \mathcal{N},$$

(3)

then $\mathcal{N}$ is an inverse $L_{n-1}$-matrix and $\mathcal{N}^\circ$ is a $L_{n-1}$-matrix, but

$$\mathcal{N}^\circ = \begin{pmatrix} A_1 & A_2^{-1} \\ 0 & A_3 \end{pmatrix} = \begin{pmatrix} A_1^{-1} & -A_1^{-1}A_2A_3^{-1} \\ 0 & A_3^{-1} \end{pmatrix}$$

is reducible. We know that all $L_{n-1}$-matrices are irreducible, this is a contradiction, so all inverse $L_{n-1}$-matrices are irreducible. This completes the proof of this theorem.

For establishing some results on eigenvalues of (inverse) $L_{n-1}$-matrices, we need the following lemma.

**Lemma 2[11]:** Assume that an $n \times n$ $A \geq 0$, then any real eigenvalue $\lambda$ of $A$ different from
$\rho(A)$ satisfies the inequality

$$\lambda \leq \rho_{\lfloor n/2 \rfloor}(A),$$

(4)

If $A$ is positive, then the inequality (4) is strict.
Lemma 3 (Gersgorin’s theorem): For any $A = (a_{ij}) \in C^{n \times n}$ and any eigenvalue $\lambda \in \sigma(A)$, there is a positive $k$ in $N=\{1,2,\ldots,n\}$ such that

$$|\lambda - a_{kk}| \leq \sum_{j \in N \setminus \{k\}} |a_{ij}|,$$

where $\sigma(A) = \{\lambda \in C : \text{det}(\lambda I - A) = 0\}$.

Theorem 4: Let $A$ be an inverse $L_{n-1}$-matrix, then $A$ has exactly one negative eigenvalue and $\det A < 0$.

Proof: Since $A^{-1} \in L_{n-1}$, then $A^{-1} = I - B (B \geq 0, \rho_{n-1}(B) \leq t < \rho(B))$, so $A^{-1}$ has a negative eigenvalue $t - \rho(B)$. According to Lemma 1, we know that $A^{-1}$ has no other negative eigenvalues, then $A$ has a negative eigenvalue $(t - \rho(B))^{-1}$, so $\det A < 0$. This completes the proof.

Theorem 5: Let $A = (a_{ij})_{n \times n}$ be an inverse $L_{n-1}$-matrix, $A^{-1} = (\overline{a_{ij}})_{n \times n}$, then

$$q(A \circ A^{-1}) > \left( |a_{ii}| - \sum_{j \in N \setminus \{i\}} |a_{ij}| + \sum_{k \in N \setminus \{i,j\}} |a_{jk}| d_k \right) |\bar{a}_{ii}|,$$

where $d_k = \frac{\sum_{j \in N \setminus \{i,k\}} |a_{ij}|}{|a_{kk}|}$ and $q(A) = \min \{|\lambda| : \lambda \in \sigma(A)\}$.

Proof: Let $\lambda \in \sigma(A \circ A^{-1})$ and $|\lambda| = q(A \circ A^{-1})$. According to lemma 3, then there exists a $i \in N$ such that

$$|\lambda - a_{ii}a_{ii}| \leq \sum_{j \in N \setminus \{i\}} |a_{ij}| \overline{a_{ij}},$$

then

$$|\lambda| \geq |a_{ii}a_{ii}| - \sum_{j \in N \setminus \{i\}} |a_{ij}| \overline{a_{ij}} - \sum_{j \in N \setminus \{i\}} |a_{ij}| \sum_{k \in N \setminus \{i,j\}} \left| a_{jk} \right| \left| d_k \right| \overline{a_{ii}}$$

$$\geq |a_{ii}a_{ii}| - \sum_{j \in N \setminus \{i\}} |a_{ij}| \sum_{j \in N \setminus \{i\}} \left| a_{jj} \right| \left| a_{ij} \right| \sum_{k \in N \setminus \{i,j\}} \left| a_{jk} \right| \left| d_k \right| \overline{a_{ii}}$$

$$= \left( |a_{ii}| - \sum_{j \in N \setminus \{i\}} \left| a_{ij} \right| \sum_{k \in N \setminus \{i,j\}} \left| a_{jk} \right| \left| d_k \right| \overline{a_{ii}} \right).$$

This completes the proof of the theorem.

Conclusions

In this paper some combinatorial results and inequalities of some classes of Z-matrices and inverse Z-matrices are given. We show that all (inverse) $L_{n-1}$ matrices are irreducible and some eigenvalue inequalities of (inverse) $L_{n-1}$ matrices. These results have wide applications in the mathematical and physical sciences.
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References