Products of cyclic fuzzy finite state machines

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Abstract. A cyclic fuzzy finite state machine is a fuzzy finite state machine which is generated by a state. In this paper, we study products of cyclic fuzzy finite state machine. We prove that two cyclic finite state machines are cyclic when the full direct product(or restricted direct product, or cascade product, or wreath product) of them is cyclic. However, products of two cyclic fuzzy finite state machines are not necessarily cyclic. We introduce the concept of strong cyclic fuzzy finite state machine, and prove that products(including full direct product, restricted direct product, cascade product and wreath product) of two strong cyclic fuzzy finite state machines are strong cyclic.

Introduction

Wee [1] first proposed the mathematical formulation of a fuzzy automaton in 1967. Malik et al. [3–6] introduced the concepts of fuzzy finite state automata, fuzzy transformation semigroup, covering, direct product, cascade product, wreath product and Cartesian composition. Almost achievements of fuzzy automata which are obtained by Malik et al. were written in [7]. On the basis of Malik et al.'s results, fuzzy automata have been further researched[8-15]. Cyclic fuzzy finite state machine which was introduced firstly by Malik et al.[6] is a class of important fuzzy finite state machine. A primary submachine[7] of a fuzzy finite state machine is a special cyclic fuzzy finite state machine. It had been proved that every fuzzy finite state machine is an union of some primary submachines of it[7]. Malik et al. had also proved that the Cartesian composition of two fuzzy finite state machines is cyclic if and only if the two fuzzy finite state machines are cyclic[6]. In this paper, we study other products of cyclic fuzzy finite state machines.

Preliminaries

Definition 2.1[5] A *fuzzy finite state machine* (ffsm) is a triple $M=(Q,X,m)$, where Q and X are finite nonempty sets and m is a fuzzy subset of $Q\times X\times Q$, i.e. $m:Q\times X\times Q\to[0,1]$.

Let X^* denote the set of all words of elements of X of finite length. The empty word is denoted by Λ, and $|u|$ denote the length of a word $u\in X^*$. Let $X^+=X^*\setminus\{\Lambda\}$.

Definition 2.2[5] Let $M=(Q,X,m)$ be a ffsm. Define the fuzzy subset m^* of $Q\times X^*\times Q$ by:

$$m^*(q,\Lambda,p)=\begin{cases}1, q=p\\ 0, q\neq p\end{cases},$$

$$m^*(q,xa,p)=\bigvee_{r\in Q}\{m^*(q,x,r)\wedge m^*(r,a,p)\} \quad , \quad \forall x\in X^*, a\in X .$$

Definition 2.3[6] Let $M=(Q,X,m)$ be a ffsm, $q_0\in Q$. If $\forall q\in Q$, $\exists u\in X^*$ such that $m^*(q_0,x,q)>0$, then q_0 is called a *generator* of M, and M is called a *cyclic* ffsm. If M is generated by q_0, then M is denoted by $M=<q_0>$.

Definition 2.4 Let $M=(Q,X,m)$ be a ffsm and $M=<q_0>$. If $\forall q\in Q$ and $\forall u\in X^+$, $m^*(q_0,u,q)>0$, then M is called a *strong cyclic* ffsm.

Definition 2.5[5] Let $M_i=(Q_i,X_i,m_i)$ be a ffsm, $i=1,2$. Let h be a function of Q_2 onto Q_1 and let x be a function of X_1 into X_2. Extend x to a function x^* of X_1^* into X_2^* by $x^*(\Lambda)=\Lambda$ and

$\forall u = x_1 x_2 \cdots x_n \in X_1^*$, $x^*(u) = x(x_1)x(x_2)\cdots x(x_n)$. Then (h,x) is called a *covering* of M_1 by M_2, written $M_1 \le M_2$, if and only if $\forall q_2 \in Q_2, q_1 \in Q_1$, and $u \in X_1^*$,

$$m_1(h(q_2),u,q_1) = \bigvee_{h(r_2)=q_1} m_2(q_2, x^*(u), r_2).$$

Definition 2.6[5] Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. $M_1 \times M_2 = (Q_1 \times Q_2, X_1 \times X_2, m_1 \times m_2)$ is called the *full direct product* of M_1 and M_2. Where

$$m_1 \times m_2((q_1,q_2),(x_1,x_2),(p_1,p_2)) = m_1(q_1,x_1,p_1) \wedge m_2(q_2,x_2,p_2)$$

$\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, (x_1,x_2) \in X_1 \times X_2$.

For $u = (a_1,b_1)(a_2,b_2)\cdots(a_n,b_n) \in (X_1 \times X_2)^*$, denote $u = (u_1,u_2)$, where $u_1 = a_1 a_2 \cdots a_n$ and $u_2 = b_1 b_2 \cdots b_n$.

Definition 2.7[5] Let $M_i = (Q_i, X, m_i)$ be a ffsm, $i = 1,2$. $M_1 \wedge M_2 = (Q_1 \times Q_2, X, m_1 \wedge m_2)$ is called the *restricted direct product* of M_1 and M_2. Where

$$m_1 \wedge m_2((q_1,q_2),x,(p_1,p_2)) = m_1(q_1,x,p_1) \wedge m_2(q_2,x,p_2)$$

$\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, \forall x \in X$.

Definition 2.8[5] Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. Let $w: Q_2 \times X_2 \to X_1$ be a function. $M_1 w M_2 = (Q_1 \times Q_2, X_2, m^w)$ is called the *cascade product* of M_1 and M_2. Where

$$m^w((q_1,q_2),x,(p_1,p_2)) = m_1(q_1, w(q_2,x), p_1) \wedge m_2(q_2,x,p_2)$$

$\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, \forall x \in X_2$.

Definition 2.9[5] Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. Let f be a function from Q_2 into X_1. $M_1 \circ M_2 = (Q_1 \times Q_2, X_1^{Q_2} \times X_2, m^\circ)$ is called the *wreath product* of M_1 and M_2. Where

$$m^\circ((q_1,q_2),(f,x),(p_1,p_2)) = m_1(q_1, f(q_2), p_1) \wedge m_2(q_2,x,p_2)$$

$\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, \forall (f,x) \in X_1^{Q_2} \times X_2$.

Lemma 2.1[5] Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. $M_1 \times M_2 = (Q_1 \times Q_2, X_1 \times X_2, m_1 \times m_2)$ is the full direct product of M_1 and M_2, then $\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2$ and $\forall (u_1,u_2) \in (X_1 \times X_2)^*$,

$$(m_1 \times m_2)^*((q_1,q_2),(u_1,u_2),(p_1,p_2)) = m_1^*(q_1,u_1,p_1) \wedge m_2^*(q_2,u_2,p_2).$$

Lemma 2.2[5] Let $M_i = (Q_i, X, m_i)$ be a ffsm, $i = 1,2$. $M_1 \wedge M_2 = (Q_1 \times Q_2, X, m_1 \wedge m_2)$ is the *re*stricted direct product of M_1 and M_2, then $\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2$ and $\forall u \in X^*$,

$$(m_1 \wedge m_2)^*((q_1,q_2),u,(p_1,p_2)) = m_1^*(q_1,u,p_1) \wedge m_2^*(q_2,u,p_2).$$

Lemma 2.3[5] Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. $M_1 w M_2 = (Q_1 \times Q_2, X_2, m^w)$ is the cascade product of M_1 and M_2. Then $\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, \forall u = a_1 a_2 \cdots a_n \in X_2^*$ where $a_1, a_2, \cdots, a_n \in X_2$,

$$m^{w*}((q_1,q_2),u,(p_1,p_2)) =$$

$$\vee\{m_1^*(q_1, w(q_2,a_1)w(q_2^{(1)},a_2)\cdots w(q_2^{(n-1)},a_n), p_1) \wedge m_2(q_2,a_1,q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)},a_n,p_2) \mid q_2^{(1)}, \cdots, \in q_2^{(n-1)} \in Q_2\}.$$

Lemma 2.4[5] Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. $M_1 \circ M_2 = (Q_1 \times Q_2, X_1^{Q_2} \times X_2, m^\circ)$ is the wreath product of M_1 and M_2. Then $\forall (q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2$, and $\forall (f_1,a_1)(f_2,a_2)\cdots(f_n,a_n) \in (X_1^{Q_2} \times X_2)^*$,

$$m^{\circ *}((q_1,q_2),(f_1,a_1)(f_2,a_2)\cdots(f_n,a_n),(p_1,p_2)) =$$

$$\vee\{m_1^*(q_1, f_1(q_2) f_2(q_2^{(1)})\cdots f_n(q_2^{(n-1)}), p_1) \wedge m_2(q_2,a_1,q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)},a_n,p_2) \mid q_2^{(i)} \in Q_2, i = 1,2,\cdots,n-1\}$$

Main results

Theorem 3.1 Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$ and let (h,x) be a covering of M_1 by M_2. If M_2 is cyclic and x is surjective, then M_1 is cyclic.

Proof. Let $M_2 = \langle p_0 \rangle$. Since h surjective, $\forall q_1 \in Q_1, \exists q_2 \in Q_2$ such that $h(q_2) = q_1$. Since $M_2 = \langle p_0 \rangle$, there exists $u_2 \in X_2^*$ such that $m_2(p_0,u_2,q_2) > 0$. Since x is surjective, there exists $u_1 \in X_1^*$ such that $x^*(u_1) = u_2$. By $M_1 \le M_2$,

$$m_1(h(p_0),u_1,q_1) = \bigvee_{h(r_2)=q_1} m_2(p_0, x^*(u_1), r_2) = \bigvee_{h(r_2)=q_1} m_2(p_0,u_2,r_2) \ge m_2(p_0,u_2,q_2) > 0.$$

Thus M_1 is generated by $h(p_0)$, i.e. M_1 is cyclic.

Theorem 3.2 Let $M_i=(Q_i,X_i,m_i)$ be a ffsm, $i=1,2$. let $M_1\times M_2=(Q_1\times Q_2,X_1\times X_2,m_1\times m_2)$ be the full direct product of M_1 and M_2. If $M_1\times M_2$ is cyclic , then M_1 and M_2 are cyclic.

Proof. Let $M_1\times M_2=<(q_0,p_0)>$. Then $\forall q_1\in Q_1$ and $\forall q_2\in Q_2$, $\exists(u_1,u_2)\in(X_1\times X_2)^*$ such that

$$(m_1\times m_2)^*((q_0,p_0),(u_1,u_2),(q_1,q_2))>0.$$

Thus $m_1^*(q_0,u_1,q_1)\wedge m_2^*(p_0,u_2,q_2)>0$, and then $m_1^*(q_0,u_1,q_1)>0$ and $m_2^*(p_0,u_2,q_2)>0$. Hence $M_1=<q_0>$ and $M_2=<p_0>$, i.e. M_1 and M_2 are cyclic.

Remark3.1 $M_1\times M_2$ is not necessarily cyclic when M_1 and M_2 are cyclic.

Example 3.1 Let $M_1=(Q_1,X_1,m_1)$, where $Q_1=\{q_1,p_1\},X_1=\{a\}$ and

$$m_1(q_1,a,q_1)=0,m_1(q_1,a,p_1)=0.1,\ m_1(p_1,a,q_1)=0.2,m_1(p_1,a,p_1)=0.$$

Let $M_2=(Q_2,X_2,m_2)$, where $Q_2=\{q_2,p_2\}$, $X_2=\{b\}$ and

$$m_2(q_2,b,q_2)=0,\ m_2(q_2,b,p_2)=0.2,m_2(p_2,b,q_2)=0.3,m_2(p_2,b,p_2)=0.$$

Then $M_1\times M_2=(Q_1\times Q_2,X_1\times X_2,m_1\times m_2)$, where $Q_1\times Q_2=\{(q_1,q_2),(q_1,p_2),(p_1,q_2),(p_1,p_2)\}$, $X_1\times X_2=\{(a,b)\}$ and

$$m_1\times m_2((q_1,q_2),(a,b),(q_1,q_2))=0,\ m_1\times m_2((q_1,p_2),(a,b),(q_1,p_2))=0,$$
$$m_1\times m_2((p_1,q_2),(a,b),(p_1,q_2))=0,\ m_1\times m_2((p_1,p_2),(a,b),(p_1,p_2))=0,$$
$$m_1\times m_2((q_1,q_2),(a,b),(q_1,p_2))=0,\ m_1\times m_2((q_1,p_2),(a,b),(q_1,q_2))=0,$$
$$m_1\times m_2((q_1,q_2),(a,b),(p_1,q_2))=0,\ m_1\times m_2((p_1,q_2),(a,b),(q_1,q_2))=0,$$
$$m_1\times m_2((p_1,q_2),(a,b),(p_1,p_2))=0,\ m_1\times m_2((p_1,p_2),(a,b),(p_1,q_2))=0,$$
$$m_1\times m_2((p_1,p_2),(a,b),(q_1,p_2))=0,\ m_1\times m_2((q_1,p_2),(a,b),(p_1,p_2))=0,$$
$$m_1\times m_2((q_1,q_2),(a,b),(p_1,p_2))=0.1,\ m_1\times m_2((p_1,p_2),(a,b),(q_1,q_2))=0.2,$$
$$m_1\times m_2((q_1,p_2),(a,b),(p_1,q_2))=0.1,\ m_1\times m_2((p_1,q_2),(a,b),(q_1,p_2))=0.2.$$

$M_1=<q_1>=<p_1>$ and $M_2=<q_2>=<p_2>$ are cyclic , but $M_1\times M_2$ is not cyclic.

Theorem 3.3 Let $M_i=(Q_i,X_i,m_i)$ be a ffsm, $i=1,2$. Let $M_1=<q_0>$ and $M_2=<p_0>$. If M_1 and M_2 are strong cyclic, then $M_1\times M_2$ is strong cyclic.

Proof. $\forall(q_1,q_2)\in Q_1\times Q_2$, $\forall(u_1,u_2)\in(X_1\times X_2)^+$, since M_1 and M_2 are strong cyclic, $m_1^*(q_0,u_1,q_1)>0$ and $m_2^*(p_0,u_2,q_2)>0$. Then $m_1^*(q_0,u_1,q_1)\wedge m_2^*(p_0,u_2,q_2)>0$. Thus $(m_1\times m_2)^*((q_0,p_0),(u_1,u_2),(q_1,q_2))>0$. Hence $M_1\times M_2=<(q_0,p_0)>$ and $M_1\times M_2$ is strong cyclic.

Theorem 3.4 Let $M_i=(Q_i,X,m_i)$ be a ffsm, $i=1,2$ and let $M_1\wedge M_2=(Q_1\times Q_2,X,m_1\wedge m_2)$ be the restricted direct product of M_1 and M_2. If $M_1\wedge M_2$ is cyclic , then M_1 and M_2 are cyclic.

Proof. Let $M_1\wedge M_2=<(q_0,p_0)>$. Then $\forall q_1\in Q_1$ and $\forall q_2\in Q_2$, $\exists u\in X^*$ such that

$$(m_1\times m_2)^*((q_0,p_0),u,(q_1,q_2))>0.$$

Thus $m_1^*(q_0,u,q_1)\wedge m_2^*(p_0,u,q_2)>0$, and then $m_1^*(q_0,u_1,q_1)>0$ and $m_2^*(p_0,u_2,q_2)>0$. Hence $M_1=<q_0>$ and $M_2=<p_0>$, i.e. M_1 and M_2 are cyclic.

Remark3.2 $M_1\wedge M_2$ is not necessarily cyclic when M_1 and M_2 are cyclic.

Example 3.2 Let $M_1=(Q_1,X,m_1)$, $M_2=(Q_2,X,m_2)$, where $Q_1=\{q_1,p_1\}$, $Q_2=\{q_2,p_2\}$, $X=\{a\}$ and

$$m_1(q_1,a,q_1)=0,m_1(q_1,a,p_1)=0.1,m_1(p_1,a,q_1)=0.2,m_1(p_1,a,p_1)=0,$$
$$m_2(q_2,a,q_2)=0,m_2(q_2,a,p_2)=0.2,m_2(p_2,a,q_2)=0.3,m_2(p_2,a,p_2)=0.$$

Similarly to Example 3.1, M_1 and M_2 are cyclic, but $M_1\wedge M_2$ is not cyclic.

Theorem 3.5 Let $M_i=(Q_i,X,m_i)$ be a ffsm, $i=1,2$. If M_1 and M_2 are strong cyclic, then $M_1\wedge M_2$ is strong cyclic.

Proof. Let $M_1=<q_0>$ and $M_2=<p_0>$. $\forall(q_1,q_2)\in Q_1\times Q_2$, $\forall u\in X^+$, since M_1 and M_2 are strong cyclic, $m_1^*(q_0,u,q_1)>0$ and $m_2^*(p_0,u,q_2)>0$. Thus $m_1^*(q_0,u,q_1)\wedge m_2^*(p_0,u,q_2)>0$, and then

$$(m_1\wedge m_2)^*((q_0,p_0),u,(q_1,q_2))>0.$$

Hence $M_1 \wedge M_2 = <(q_0, p_0)>$ and $M_1 \wedge M_2$ is strong cyclic.

Theorem 3.6 Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$ and let $M_1 w M_2 = (Q_1 \times Q_2, X_2, m^w)$ be the cascade product of M_1 and M_2. If $M_1 w M_2$ is cyclic , then M_1 and M_2 are cyclic.

Proof. Let $M_1 w M_2 = <(q_0, p_0)>$. Then $\forall p_1 \in Q_1$ and $\forall p_2 \in Q_2$, $\exists u_2 \in X_2^*$ such that $m^{w*}((q_0, p_0), u_2, (p_1, p_2)) > 0$. Let $u_2 = a_1 a_2 \cdots a_n \in X_2^*$, then by Lemma 2.3,

$$m^{w*}((q_0, p_0), u_2, (p_1, p_2)) =$$

$$\vee\{m_1^*(q_0, w(p_0, a_1) w(q_2^{(1)}, a_2) \cdots w(q_2^{(n-1)}, a_n), p_1) \wedge m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) \mid q_2^{(1)}, \cdots, q_2^{(n-1)} \in Q_2\}.$$

Thus $\exists q_2^{(1)}, q_2^{(2)}, \cdots, q_2^{(n-1)} \in Q_2$ such that

$$m_1^*(q_0, w(p_0, a_1) w(q_2^{(1)}, a_2) \cdots w(q_2^{(n-1)}, a_n), p_1) \wedge m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0,$$

and then

$$m_1^*(q_0, w(p_0, a_1) w(q_2^{(1)}, a_2) \cdots w(q_2^{(n-1)}, a_n), p_1) > 0, \quad m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0.$$

Hence $M_1 = <q_0>$, i.e. M_1 is cyclic.

Since

$$m_2(p_0, u, p_2) \geq m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0,$$

$M_2 = <p_0>$, i.e. M_2 is cyclic.

Remark3.3 $M_1 w M_2$ is not necessarily cyclic when M_1 and M_2 are cyclic.

Example 3.3 Let $M_1 = (Q_1, X_1, m_1)$ and $M_2 = (Q_2, X_2, m_2)$ be define as Example 3.1. Let

$$w: Q_2 \times X_2 \to X_1; \ (q_2, b) \mapsto a, (p_2, b) \mapsto a.$$

Then for $M_1 w M_2 = (Q_1 \times Q_2, X_2, m^w)$,

$$m^w((s_1, s_2), b, (r_1, r_2)) = m_1(s_1, w(s_2, b), r_1) \wedge m_2(s_2, b, r_2) = m_1(s_1, a, r_1) \wedge m_2(s_2, b, r_2),$$

$\forall (s_1, s_2), (r_1, r_2) \in Q_1 \times Q_2$. Thus $M_1 w M_2$ is defined essentially the same as $M_1 \times M_2$ in Example 3.1, hence $M_1 w M_2$ is not cyclic.

Theorem 3.7 Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. If M_1 and M_2 are strong cyclic, then $M_1 w M_2$ is strong cyclic.

Proof. Let $M_1 = <q_0>$ and $M_2 = <p_0>$. $\forall (q_1, q_2) \in Q_1 \times Q_2$, $\forall u_2 = a_1 a_2 \cdots a_n \in X_2^+$, since M_2 is strong cyclic, $m_2^*(p_0, u_2, q_2) > 0$. And then $\exists q_2^{(1)}, q_2^{(2)}, \cdots, q_2^{(n-1)} \in Q_2$ such that $m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0$. Since M_1 is strong cyclic, $m_1^*(q_0, w(p_0, a_1) w(q_2^{(1)}, a_2) \cdots w(q_2^{(n-1)}, a_n), p_1) > 0$. Thus

$$m_1^*(q_0, w(p_0, a_1) w(q_2^{(1)}, a_2) \cdots w(q_2^{(n-1)}, a_n), p_1) \wedge m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0.$$

Then by Lemma 2.3,

$$m^{w*}((q_0, p_0), u_2, (p_1, p_2)) \geq m_1^*(q_0, w(p_0, a_1) w(q_2^{(1)}, a_2) \cdots w(q_2^{(n-1)}, a_n), p_1) \wedge m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0.$$

Hence $M_1 w M_2 = <(q_0, p_0)>$ and $M_1 w M_2$ is strong cyclic.

Theorem 3.8 Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$ and let $M_1 \circ M_2 = (Q_1 \times Q_2, X_1^{Q_2} \times X_2, m^\circ)$ be the wreath product of M_1 and M_2. If $M_1 \circ M_2$ is cyclic , then M_1 and M_2 are cyclic.

Proof. Let $M_1 \circ M_2 = <(q_0, p_0)>$. Then $\forall p_1 \in Q_1$ and $\forall p_2 \in Q_2$, $\exists u \in (X_1^{Q_2} \times X_2)^*$ such that

$$m^{w*}((q_0, p_0), u, (p_1, p_2)) > 0.$$

Let $u = (f_1, a_1)(f_2, a_2) \cdots (f_n, a_n)$, then by Lemma 2.4,

$$m^{\circ *}((q_0, p_0), (f_1, a_1)(f_2, a_2) \cdots (f_n, a_n), (p_1, p_2)) =$$

$$\vee\{m_1^*(q_0, f_1(p_0) f_2(q_2^{(1)}) \cdots f_n(q_2^{(n-1)}), p_1) \wedge m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) \mid q_2^{(i)} \in Q_2, i = 1,2,\cdots, n-1\}.$$

Thus $\exists q_2^{(1)}, q_2^{(2)}, \cdots, q_2^{(n-1)} \in Q_2$ such that

$$m_1^*(q_0, f_1(p_0) f_2(q_2^{(1)}) \cdots f_n(q_2^{(n-1)}), p_1) \wedge m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0,$$

and then

$$m_1^*(q_0, f_1(p_0) f_2(q_2^{(1)}) \cdots f_n(q_2^{(n-1)}), p_1) > 0, \quad m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0.$$

Hence $M_1 = <q_0>$, i.e. M_1 is cyclic. Since

$$m_2(p_0, a_1 a_2 \cdots a_n, p_2) \geq m_2(p_0, a_1, q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)}, a_n, p_2) > 0,$$

$M_2 =< p_0 >$, i.e. M_2 is cyclic.

Remark3.4 $M_1 \circ M_2$ is not necessarily cyclic when M_1 and M_2 are cyclic.

Example 3.4 Let $M_1 = (Q_1, X_1, m_1)$ and $M_2 = (Q_2, X_2, m_2)$ be define as Example 3.1. Since

$$m^{\circ}((s_1, s_2),(f,b),(r_1,r_2)) = m_1(s_1, f(s_2), r_1) \wedge m_2(s_2,b,r_2) = m_1(s_1,a,r_1) \wedge m_2(s_2,b,r_2),$$

$\forall (s_1,s_2),(r_1,r_2) \in Q_1 \times Q_2$ and $\forall (f,b) \in X_1^{Q_2} \times X_2$, $M_1 \circ M_2$ is defined essentially the same as $M_1 \times M_2$ in Example 3.1, hence $M_1 \circ M_2$ is not cyclic.

Theorem 3.9 Let $M_i = (Q_i, X_i, m_i)$ be a ffsm, $i = 1,2$. If M_1 and M_2 are strong cyclic, then $M_1 \circ M_2$ is strong cyclic.

Proof. Let $M_1 =< q_0 >$ and $M_2 =< p_0 >$. $\forall (q_1,q_2) \in Q_1 \times Q_2$ and $\forall u = (f_1,a_1)(f_2,a_2)\cdots(f_n,a_n) \in (X_1^{Q_2} \times X_2)^*$, let $u_2 = a_1 a_2 \cdots a_n$. Since M_2 is strong cyclic, $m_2^*(p_0,u_2,q_2) > 0$. And then $\exists q_2^{(1)}, q_2^{(2)}, \cdots, q_2^{(n-1)} \in Q_2$ such that $m_2(p_0,a_1,q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)},a_n,p_2) > 0$. Since M_1 is strong cyclic, $m_1^*(q_0, f_1(p_0) f_2(q_2^{(1)}) \cdots f_n(q_2^{(n-1)}), p_1) > 0$.
Thus

$$m_1^*(q_0, f_1(p_0) f_2(q_2^{(1)}) \cdots f_n(q_2^{(n-1)}), p_1) \wedge m_2(p_0,a_1,q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)},a_n,p_2) > 0.$$

By Lemma 2.4,

$$m^{\circ *}((q_0,p_0),(f_1,a_1)(f_2,a_2)\cdots(f_n,a_n),(p_1,p_2)) \geq$$
$$m_1^*(q_0, f_1(p_0) f_2(q_2^{(1)}) \cdots f_n(q_2^{(n-1)}), p_1) \wedge m_2(p_0,a_1,q_2^{(1)}) \wedge \cdots \wedge m_2(q_2^{(n-1)},a_n,p_2) > 0.$$

Hence $M_1 \circ M_2 = < (q_0,p_0) >$ and $M_1 \circ M_2$ is strong cyclic.

Conclusion

In this paper, we obtain the following conclusions.

(1) we prove that M_1 and M_2 are cyclic when the full direct product(or restricted direct product, or cascade product, or wreath product) of M_1 and M_2 is cyclic.

(2) we give counter examples to show that products of M_1 and M_2 are not cyclic when M_1 and M_2 are cyclic.

(3) By introducing the concept of strong cyclic fuzzy finite automata, we prove that products (full direct product, restricted direct product, cascade product and wreath product) of M_1 and M_2 are strong cyclic when M_1 and M_2 are strong cyclic.

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