Algorithms for the Detection of a Signal with Unknown Amplitude and Duration against White Noise

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Abstract—We synthesized the quasi-likelihood and maximum likelihood algorithms for the detection of a free-form signal with unknown amplitude and duration against Gaussian white noise. We illustrated the methods for calculation of the characteristics of their operating effectiveness and also we found the analytical expressions for false-alarm and missing probabilities of the considered detectors.

Keywords—free-form signal; detection; maximum likelihood method; parametrical prior uncertainty; false-alarm and missing probabilities

I. INTRODUCTION

The problem of the detection of information signals with unknown power and time parameters is vital in radio physics and radio engineering. So, in works [1, 2] the reception of a rectangular pulse with unknown duration against white noise was considered. In [3] algorithms of reception of a free-form pulse with unknown duration were studied. However, in a number of practical applications the received signal amplitude is also unknown. In work [4] algorithms of reception of a rectangular pulse with unknown duration and amplitude were synthesized. The asymptotic expressions for detection error probabilities, and for distribution functions and variances of joint maximum likelihood estimates of a signal amplitude and duration were found. In the present work we carry out the synthesis and analysis of the algorithms for the detection of a free-form pulse with unknown duration and amplitude.

II. PROBLEM STATEMENT

We suppose that the signal

$$s(t, a_0, \tau_0) = \begin{cases} a_0 f(t), & 0 \leq t \leq \tau_0, \\ 0, & t < 0, t > \tau_0 \end{cases} \quad (1)$$

is to be detected against Gaussian white noise $n(t)$ with one-sided spectral density N_0 . Thus, the realization $\xi(t)$ is observed over the time interval $[0, T]$, and it can be by noise

only: $\xi(t) = n(t)$ or by additive mix of a signal and noise: $\xi(t) = s(t, a_0, \tau_0) + n(t)$. In Eq. (1) a_0 , τ_0 are the designations of the signal amplitude and duration accordingly, and $f(t)$ is the function describing the form of a signal.

Let us assume that the signal duration possesses the value within a prior interval $\tau \in [T_1, T_2]$, and function $f(t)$ is satisfied to a condition $\max f(t) = 1$, $t \in [0, T_2]$. With observable realization $\xi(t)$ and prior information, it is necessary to make the decision on presence or absence of a useful signal (1).

If the useful signal (1) amplitude and duration are a priori known, then it is possible to apply optimal detection algorithm [2, 5, 6] according to which a receiver forms the logarithm of the functional of likelihood ratio (FLR)

$$L_0 = \frac{2a_0}{N_0} \int_0^{\tau_0} \xi(t) f(t) dt - \frac{a_0^2}{N_0} \int_0^{\tau_0} f^2(t) dt \quad (2)$$

The decision on signal presence or absence is made as a result of the comparison of the generated value (2) with a threshold chosen according to the accepted optimality criterion. However, under unknown signal parameters a_0 and τ_0 we have a prior uncertainty concerning amplitude and duration. In this case the logarithm of FLR is a function of two variables [2, 6]

$$L(a, \tau) = \frac{2a}{N_0} \int_0^{\tau} \xi(t) f(t) dt - \frac{a^2}{N_0} \int_0^{\tau} f^2(t) dt, \quad (3)$$

depending on current values a , τ of unknown parameters a_0 , τ_0 . Correspondingly, a number of detection algorithms (probably, nonoptimal) can be synthesized by substituting some specific levels of a and τ in Eq. (3). These levels can be fixed

or they can be determined by the realization of the observable data, as it is considered below.

III. QUASI-LIKELIHOOD DETECTION ALGORITHM

The simplest way for the overcoming of the prior uncertainty concerning amplitude and duration is the application of the quasi-likelihood (QL) detection algorithm. QL receiver forms the logarithm of FLR for some expected (predictable) amplitude a^* and duration $\tau^* \in [T_1, T_2]$, i.e.

$$L_1 = L(a^*, \tau^*) = \frac{2a^*}{N_0} \int_0^{\tau^*} \xi(t) f(t) dt - \frac{a^{*2}}{N_0} \int_0^{\tau^*} f^2(t) dt \quad (4)$$

The value (4) is compared to a threshold h and a decision is made on presence or absence of a signal in observable realization.

In order to determine the QL algorithm performance, we find the analytical expressions for false-alarm and missing probabilities [2, 5]. It is obvious that amplitude and duration ignorance influences detection efficiency. Therefore, we put into consideration the values characterizing the amplitude detuning $\Delta_a = a^*/a_0$, $\delta_a = (a^* - a_0)/a_0 = \Delta_a - 1$ and duration detuning $\Delta_\tau = \tau^*/\tau_0$, $\delta_\tau = (\tau^* - \tau_0)/\tau_0 = \Delta_\tau - 1$, so

$$a^* = a_0(1 + \delta_a) = a_0\Delta_a, \quad \tau^* = \tau_0(1 + \delta_\tau) = \tau_0\Delta_\tau. \quad (5)$$

Substituting Eqs. (5) in Eq. (4) we have

$$L_1 = \frac{2a_0}{N_0} \Delta_a \int_0^{\tau_0\Delta_\tau} \xi(t) f(t) dt - \frac{a_0^2\Delta_a^2}{N_0} \int_0^{\tau_0\Delta_\tau} f^2(t) dt.$$

Let us designate $L_1 = L_{11}$ as the logarithm of FLR, if a useful signal is present in the received realization, and $L_1 = L_{10}$ – if a useful signal is absent. The random variables L_{11} , L_{10} are Gaussian being linear transformations from Gaussian white noise. Therefore, it is enough to find their mathematical expectations $S_{11} = \langle L_{11} \rangle$, $S_{10} = \langle L_{10} \rangle$ and dispersions

$D_{11} = \langle (L_{11} - S_{11})^2 \rangle$, $D_{10} = \langle (L_{10} - S_{10})^2 \rangle$ for full statistical description. Carrying out averaging over all the realizations of the observable data we obtain

$$S_{11} = z_0^2 \Delta_a [1 + d(\min(1, \Delta_\tau))] - z_0^2 \Delta_a^2 [1 + d(\Delta_\tau)]/2,$$

$$S_{10} = -z_0^2 \Delta_a^2 [1 + d(\Delta_\tau)]/2, \quad D_{11} = D_{10} = D = z_0^2 \Delta_a^2 [1 + d(\Delta_\tau)], \quad (6)$$

where $z_0^2 = (2a_0^2/N_0) \int_0^{\tau_0} f^2(t) dt$ is the output signal-to-noise ratio (SNR) [2, 6] for optimal receiver (2), and

$d(x) = \int_{\tau_0}^{\tau_0+x} f^2(t) dt / \int_0^{\tau_0} f^2(t) dt$ is the generalized detuning on duration.

Using statistical characteristics (6) of the random variables L_{11} , L_{10} we write down exact expressions for false-alarm probability:

$$\alpha_1 = P[L_{10} > h] = 1 - P[L_{10} < h] = 1 - \Phi \left[\frac{h/z_0 \Delta_a \sqrt{1+d(\Delta_\tau)} + (z_0 \Delta_a/2) \sqrt{1+d(\Delta_\tau)}}{\sqrt{1+d(\Delta_\tau)}} \right] \quad (7)$$

and missing probability:

$$\beta_1 = P[L_{11} < h] = F_{11}(h) = \Phi \left[\frac{h/z_0 \Delta_a \sqrt{1+d(\Delta_\tau)} - z_0(1+d(\min(1, \Delta_\tau))) / \sqrt{1+d(\Delta_\tau)} + (z_0 \Delta_a/2) \sqrt{1+d(\Delta_\tau)}}{\sqrt{1+d(\Delta_\tau)}} \right]. \quad (8)$$

Here $\Phi(x) = \int_{-\infty}^x \exp(-t^2/2) dt / \sqrt{2\pi}$ is the probability integral.

If expected amplitude and duration coincide with their true values, i.e. $\tau_0 = \tau^*$ and $a_0 = a^*$, then QL detector coincides with optimal detector for a priori known signal, which possesses characteristics [5]

$$\alpha_0 = 1 - \Phi(h/z_0 + z_0/2), \quad \beta_0 = \Phi(h/z_0 - z_0/2) \quad (9)$$

Eqs. (7), (8) and (9) allow us to define the losses in efficiency of QL detector (4) in comparison with optimal detector (2) owing to a departure of a^* from a_0 and τ^* from τ_0 .

IV. AMPLITUDE ADAPTATION ALGORITHM

In order to increase detection efficiency, QL amplitude adaptation algorithm can be used according to which the logarithm of FLR is formed for expected duration τ^* and its maximization on amplitude is carried out:

$$L_2 = \max_a L(a, \tau^*) = \max_a L_2(a) \quad (10)$$

The decision on signal presence or absence is made by comparing the value (10) with a threshold h . Maximization of the logarithm of FLR (3) on amplitude can be carried out analytically. Really, it is easily seen that the function $L_2(a) = L(a, \tau^*)$ tops, if

$$a = \int_0^{\tau^*} \xi(t) f(t) dt / \int_0^{\tau^*} f^2(t) dt \quad (11)$$

Substituting Eq. (11) in Eq. (10) we obtain the following expression for the decision statistics:

$$L_2 = \left(\int_0^{\tau^*} \xi(t) f(t) dt \right) / \sqrt{N_0 \int_0^{\tau^*} f^2(t) dt} \quad (12)$$

While analyzing the synthesized algorithm, we find the distribution function of a random variable (12). We designate $L_2 = L_{21}$ in case of the signal (1) presence, and $L_2 = L_{20}$, when the signal (1) is absent. In Eq. (12) the value

$\kappa = \int_0^{\tau^*} \xi(t) f(t) dt / \sqrt{N_0 \int_0^{\tau^*} f^2(t) dt}$ is squared, which is Gaussian random one with mathematical expectation $m = 0$ (when the signal is absent) or $m = a_0 \int_0^{\min(\tau_0, \tau^*)} \xi(t) f^2(t) dt / \sqrt{N_0 \int_0^{\tau^*} f^2(t) dt}$ (when the signal is present) and dispersion $\sigma^2 = 1/2$. Hence, for the distribution function of the random variable L_2 (12) we have

$$F_{L_2}(y) = P[L_2 < y] = P[-\sqrt{y} < \kappa < \sqrt{y}] = F_{\kappa}(\sqrt{y}) - F_{\kappa}(-\sqrt{y}) \quad (13)$$

where $F_{\kappa}(y)$ is distribution function of the Gaussian random variable κ . Using Eq. (13) we now write down exact expressions for false-alarm probability:

$$\alpha_2 = 1 - P[L_{20} < h] = 2[1 - \Phi(2\sqrt{h})] \quad (14)$$

and missing probability:

$$\begin{aligned} \beta_2 = P[L_{21} < h] &= \Phi \left[2\sqrt{h} - \frac{z_0(1 + d(\min(1, \Delta_{\tau})))}{\sqrt{(1 + d(\Delta_{\tau}))/2}} \right] + \\ &+ \Phi \left[2\sqrt{h} + \frac{z_0(1 + d(\min(1, \Delta_{\tau})))}{\sqrt{(1 + d(\Delta_{\tau}))/2}} \right] - 1. \end{aligned} \quad (15)$$

Comparing Eqs. (14), (15) and Eqs. (7), (8) we can determine a detection efficiency gain resulting from the amplitude adaptation realization.

V. DURATION ADAPTATION ALGORITHM

Other way of improvement of the detection quality in comparison with Eq. (4) is the use of QL detection algorithm with adaptation on duration. The receiver forms the logarithm of FLR (3) for expected amplitude, while maximization on duration is carried out

$$L_3 = \max_{\tau} L(a^*, \tau) = \max_{\tau} L_3(\tau) \quad (16)$$

The decision on signal presence or absence is made as a result of comparison of value (16) with a threshold h . We designate

$$L_3(\tau) = L(a^*, \tau) = \frac{2a^*}{N_0} \int_0^{\tau} \xi(t) f(t) dt - \frac{a^{*2}}{N_0} \int_0^{\tau} f^2(t) dt \quad (17)$$

Let $L_3(\tau) = L_{30}(\tau)$, if the signal (1) is absent, and $L_3(\tau) = L_{31}(\tau)$, if the signal (1) is present in the received realization. According to Eq. (17), the random processes $L_{30}(\tau)$, $L_{31}(\tau)$ are Gaussian. Therefore, it is enough to find their mathematical expectations and covariance functions for the full statistical description. Carrying out averaging over all the realizations of the observable data we get

$$S_{30}(\tau) = \langle L_{30}(\tau) \rangle = -\Delta_a^2 q(\tau)/2,$$

$$S_{31}(\tau) = \langle L_{31}(\tau) \rangle = \Delta_a q(\min(\tau, \tau_0)) - \Delta_a^2 q(\tau)/2,$$

$$\begin{aligned} K(\tau_1, \tau_2) &= \langle [L_{3i}(\tau_1) - \langle L_{3i}(\tau_1) \rangle][L_{3i}(\tau_2) - \langle L_{3i}(\tau_2) \rangle] \rangle = \\ &= \Delta_a^2 q(\min(\tau_1, \tau_2)), \quad i = 0, 1. \end{aligned}$$

Here $q(\tau) = (2a_0^2/N_0) \int_0^{\tau} f^2(t) dt$ is output SNR for a signal with duration τ .

In Eq. (17) we move to a new variable $\lambda = q(\tau)$. As function $q(\tau)$ is nonnegative and monotonic, the following equality $q(\min(\tau_1, \tau_2)) = \min(q(\tau_1), q(\tau_2))$ is valid. Then, we can present the decision statistics (17) as function of the variable λ in the form of

$$L_{31}(\tau) = L_{31}[\tau(\lambda)] = \mu_{31}(\lambda) = \Delta_a \min(\lambda, \lambda_0) - \Delta_a^2 \lambda/2 + v(\lambda)$$

with the signal (1) present and

$$L_{30}(\tau) = L_{30}[\tau(\lambda)] = \mu_{30}(\lambda) = -\Delta_a^2 \lambda/2 + v(\lambda)$$

with the signal (1) absent. Here $\lambda_0 = q(\tau_0)$, and $v(\lambda)$ is Gaussian random process with zero mathematical expectation and covariance function

$$B(\lambda_1, \lambda_2) = \Delta_a^2 \min(\lambda_1, \lambda_2), \quad (18)$$

and $\tau(\lambda)$ is determined from solution of equation $q(\tau) = \lambda$.

By definition the false-alarm probability α_3 can be expressed in terms of the probability $P_0(h)$ of threshold h uncrossing by the random process $\mu_{30}(\lambda)$, that is $\alpha_3 = 1 - P\{-\infty < \mu_{30}(\lambda) < h, \lambda \in [\Lambda_1, \Lambda_2]\} = 1 - P_0(h)$

Similarly, the missing probability is equal to probability $P_1(h)$ of the threshold h uncrossing by the random process $\mu_{31}(\lambda)$: $\beta_3 = P\{-\infty < \mu_{31}(\lambda) < h, \lambda \in [\Lambda_1, \Lambda_2]\} = P_1(h)$. Here it is designated as

$$P_i(h) = P\{-\infty < \mu_{3i}(\lambda) < h, \lambda \in [\Lambda_1, \Lambda_2]\} \quad (19)$$

In order to find the functions (19), we use the procedure stated in [7, 8]. According to Eq. (18), the random process $\mu_{3i}(\lambda)$ is Markov one with drift k_{1i} and diffusion k_{2i} coefficients

$$k_{1i} = \begin{cases} i\Delta_a - \Delta_a^2/2, & \lambda \leq \lambda_0, \\ -\Delta_a^2/2, & \lambda > \lambda_0, \end{cases} \quad k_{2i} = \Delta_a^2, \quad i = 0, 1 \quad (20)$$

Therefore, we can write down [7, 8]

$$P_i(h) = P\{\mu_{3i}(\lambda) > 0, \lambda \in [\Lambda_1, \Lambda_2]\} = \int_0^\infty W_i(y, \Lambda_2) dy \quad (21)$$

Here $W_i(y, \lambda)$ is a solution of the direct Fokker-Planck-Kolmogorov equation [7-9]

$$\frac{\partial W_i(y, \lambda)}{\partial \lambda} + \frac{\partial}{\partial y} [k_{1i} W_i(y, \lambda)] - \frac{1}{2} \frac{\partial^2}{\partial y^2} [k_{2i} W_i(y, \lambda)] = 0 \quad (22)$$

with boundary conditions $W_i(y=0, \lambda) = W_i(y=\infty, \lambda) = 0$ and starting condition

$$W_i(y, \lambda)|_{\lambda=\Lambda_1} = \frac{1}{\Delta_a \sqrt{2\pi\Lambda_1}} \exp\left[-\frac{(y-h+i\Delta_a\Lambda_1-\Delta_a^2\Lambda_1/2)^2}{2\Delta_a^2\Lambda_1}\right]$$

Applying a reflection method with sign inversion, we find the solution of Eq. (22) with coefficients (20) separately for the cases $\lambda \in [\Lambda_1, \lambda_0]$ and $\lambda \in [\lambda_0, \Lambda_2]$, as it is described in [7]. Further, following [8] and substituting the found solutions in Eq. (21) we obtain expressions for false-alarm and missing probabilities in the form of

$$\alpha_3 = 1 - \frac{1}{\Delta_a \sqrt{2\pi\Lambda_1}} \int_0^\infty \exp\left[-\frac{(h-\xi+\Delta_a^2\Lambda_1/2)^2}{2\Delta_a^2\Lambda_1}\right] \times \\ \times \left\{ \Phi\left(\frac{\Delta_a\sqrt{\Lambda_2-\Lambda_1}}{2} + \xi/\Delta_a\sqrt{\Lambda_2-\Lambda_1}\right) - \exp(-\xi)\Phi\left(\frac{\Delta_a\sqrt{\Lambda_2-\Lambda_1}}{2} - \xi/\Delta_a\sqrt{\Lambda_2-\Lambda_1}\right) \right\} d\xi, \quad (23)$$

$$\beta_3 = \frac{\exp\left[(\Lambda_1-\lambda_0)(1-\Delta_a/2)^2/2\right]}{2\pi\Delta_a^2\sqrt{\Lambda_1(\lambda_0-\Lambda_1)}} \int_0^\infty \int_0^\infty d\xi d\xi_1 \exp[(\xi-\xi_1) \times \\ \times (1/\Delta_a - 1/2) - (h-\xi-\Delta_a\Lambda_1+\Delta_a^2\Lambda_1/2)^2/2\Delta_a^2\Lambda_1] \times \\ \times \left\{ \Phi\left(\frac{\Delta_a\sqrt{\Lambda_2-\lambda_0}}{2} + \xi_1/\Delta_a\sqrt{\Lambda_2-\lambda_0}\right) - \exp(-\xi_1)\Phi\left(\frac{\Delta_a\sqrt{\Lambda_2-\lambda_0}}{2} - \xi_1/\Delta_a\sqrt{\Lambda_2-\lambda_0}\right) \right\} \times \\ \times \left\{ \exp\left[-(\xi_1-\xi)^2/2\Delta_a^2(\lambda_0-\Lambda_1)\right] - \exp\left[-(\xi_1+\xi)^2/2\Delta_a^2(\lambda_0-\Lambda_1)\right] \right\}. \quad (24)$$

Comparing Eqs. (23), (24) and (7), (8) we can determine detection efficiency gain due to the application of the adaptation on duration and then make the decision on the expediency of such adaptation application.

VI. MAXIMUM LIKELIHOOD DETECTION ALGORITHM

In order to improve a detection quality, it is possible to apply a maximum likelihood detection algorithm according to which the receiver carries out a maximization of the logarithm of FLR (3) on both duration and amplitude:

$$L_4 = \max_{\tau} L_4(\tau), \quad L_4(\tau) = \max_a L(a, \tau)$$

The decision on signal presence or absence is made comparing the value L_4 with a threshold h . Similarly to Eqs. (11), (12) the maximization of the logarithm of FLR (3) on variable a can be carried out analytically. As a result we have

$$L_4(\tau) = \frac{1}{N_0} \left(\int_0^\tau \xi(t) f(t) dt \right)^2 \bigg/ \int_0^\tau f^2(t) dt \quad (25)$$

Eq. (25) defines the detector structure. The detector should form the random process (25) for all possible duration values and find its maximum. Comparing this maximum with a threshold the decision is made on presence or absence of the useful signal.

Let us find the detection performances for the algorithm working on the basis of the Eq. (25). For this purpose we consider the random process $M(\tau) = \int_0^\tau \xi(t) f(t) dt$, which is Gaussian one with the zero mathematical expectation in the signal absence and the mathematical expectation

$$S_M(\tau) = \langle M(\tau) \rangle = a_0 \int_0^{\min(\tau, \tau_0)} f^2(t) dt = a_0 \rho(\min(\tau, \tau_0))$$

in the signal presence. Here it is designated

$$\rho(\tau) = \int_0^\tau f^2(t) dt \quad (26)$$

Covariance function of the process $M(\tau)$ is of the form

$$K_M(\tau_1, \tau_2) = \langle [M(\tau_1) - \langle M(\tau_1) \rangle][M(\tau_2) - \langle M(\tau_2) \rangle] \rangle = (N_0/2)\rho(\min(\tau_1, \tau_2)).$$

We move to the new variable $r = \rho(\tau)$, $r \in [R_1, R_2]$, $R_1 = \rho(T_1)$, $R_2 = \rho(T_2)$ in Eq. (25). As the function $\rho(\tau)$ is nonnegative and monotonic, we have $\rho(\min(\tau_1, \tau_2)) = \min(\rho(\tau_1), \rho(\tau_2))$. Then, the decision statistics (25) as function of the variable r can be presented in a kind of

$$L_4(r) = \chi^2(r)/N_0r \quad (27)$$

Here $\chi(r)$ is Gaussian random process with covariance function

$$K_\chi(r_1, r_2) = (N_0/2)\min(r_1, r_2) \quad (28)$$

and also zero mathematical expectation in the signal absence and mathematical expectation

$$S_\chi(r) = a_0 \min(r, r_0), \quad r_0 = \rho(\tau_0) \quad (29)$$

in the signal presence. From Eqs. (28), (29) follows that the random process $\chi(r)$ can be presented as $\chi_0(r) = \omega(r)$ in the signal absence and $\chi_1(r) = a_0 \min(r, r_0) + \omega(r)$ in the signal presence, where $\omega(r)$ is Wiener process [9].

In order to determine the false-alarm probability for ML detector, we move to the new variable $\theta = \ln r$, $\theta \in [\theta_1, \theta_2]$, $\theta_1 = \ln R_1$, $\theta_2 = \ln R_2$ in Eq. (27). According to [9], the function $L_4(\theta) = L_4(r(\theta))$ is Markov random process which satisfies to the stochastic differential Ito's equation $dL_4(\theta) = -L_4(\theta)d\theta + \sqrt{2L_4(\theta)}d\omega(\theta)$. It is known [2] that with the increase of the threshold h and the length of the prior interval $[\theta_1, \theta_2]$ the probability $P[\max L_4(\theta) < h]$ is limited to

$$P[\max L_4(\theta) < h] \rightarrow \exp[-\sqrt{h/\pi}(\theta_2 - \theta_1)\exp(-h)].$$

Returning to the variable $r = \exp(\theta)$ in the last formula we have for large values of the threshold h and relation R_2/R_1 :

$$P[\max L_4(r) < h] \approx (R_1/R_2)^{\sqrt{h/\pi}\exp(-h)}.$$

From here we obtain for false-alarm probability: $\alpha_4 = P[\max L_4(r) > h] \approx 1 - P[L_4(r) < h]$. Taking into account that the false-alarm probability should be never-decreasing function of a threshold, we specify last formula as

$$\alpha_4 \approx \begin{cases} 1 - (R_1/R_2)^{\sqrt{h/\pi}\exp(-h)}, & h \geq 1/2, \\ 1, & h < 1/2. \end{cases} \quad (30)$$

Now, moving to the variable $\tau = \tau(r)$ in Eq. (30), where $\tau(r)$ is the solution of equation $r = \rho(\tau)$ (26), the following final asymptotic expression for the false-alarm probability can be written down

$$\alpha_4 \approx \begin{cases} 1 - \left(\int_0^{T_1} f^2(t)dt / \int_0^{T_2} f^2(t)dt \right)^{\sqrt{h/\pi}\exp(-h)}, & h \geq 1/2, \\ 1, & h < 1/2 \end{cases} \quad (31)$$

with large values of a threshold and relation T_2/T_1 .

Let us pass to the determining of the missing probability. In the signal presence at the receiver input the decision statistics (27) has the appearance

$$L_4(r) = z_0^2 \frac{\min^2(1, l)}{2l} + z_0 \frac{\min(1, l)}{l} \omega(l) + \frac{\omega^2(l)}{2l}, \quad (32)$$

where $l = r/r_0$, $r_0 = \int_0^{\tau_0} f^2(t)dt$, and z_0 is defined the same way as in Eqs. (6).

With large SNR we can neglect the last summand in Eq. (32) and write down the approximate equality

$$L_4(r) \approx z_0^2 \min^2(1, l)/2l + z_0 \min(1, l)\omega(l)/l \quad (33)$$

This function is Markov random process with drift k_1 and diffusion k_2 coefficients having the appearance of

$$k_1 \approx \frac{z_0^2}{2r_0} \begin{cases} 1, & \tau < \tau_0, \\ -1/l^2, & \tau > \tau_0, \end{cases} \quad k_2 = 2|k_1| \quad (34)$$

Markovian properties of the process (33) allow finding the auxiliary function

$$F(u, v, R) = P \left[\begin{matrix} L_4(r) < u, & L_4(r) < v \\ R_1 < r < R & R < r < R_2 \end{matrix} \right], \quad (35)$$

through which the missing probability can be easily expressed [8]

$$\beta_4 = P[\max L_4(r) < h] = F(h, h, R_2) \quad (36)$$

Following [8], for function (35) we can write down $F(u, v, R) = \int_{-\infty}^u W(x, R_2) dx$, where $W(x, R)$ is a solution of the Fokker-Planck-Kolmogorov equation (22) with coefficients (34), boundary conditions $W(x, r)|_{x=u} = 0$, $r \in [R_1, R]$ and $W(x, r)|_{x=v} = 0$, $r \in [R, R_2]$, and starting condition $W(x, r)|_{r=R_1} = W_1(x)$. Here $W_1(x)$ is the probability density of the random value $L_4(R_1)$ (33). By solving this equation, as it is described in [8], we find the function (35), which we substitute in Eq. (36) and obtain the asymptotic expression for the missing probability in the form of

$$\begin{aligned} \beta_4 \approx & \int_0^\infty d\xi_1 \int_0^h d\xi \frac{\text{ch}(z_0 \sqrt{2\rho(T_1)(h-\xi)/\rho(\tau_0)})}{\pi z_0 \sqrt{2(h-\xi)[1-\rho(T_1)/\rho(\tau_0)]}} \left\{ 1 - \right. \\ & \left. - \exp \left[- \frac{2\xi\xi_1}{z_0^2(1-\rho(T_1)/\rho(\tau_0))} \right] \right\} \left\{ \Phi \left(\frac{z_0^2[1-\rho(\tau_0)/\rho(T_2)]/2 + \xi_1}{z_0 \sqrt{1-\rho(\tau_0)/\rho(T_2)}} \right) - \right. \\ & \left. \times \exp(-\xi_1) \Phi \left(\frac{z_0^2[1-\rho(\tau_0)/\rho(T_2)]/2 - \xi_1}{z_0 \sqrt{1-\rho(\tau_0)/\rho(T_2)}} \right) \right\} \times \\ & \times \exp \left\{ \xi - h - \frac{z_0^2 \rho(T_1)}{2\rho(\tau_0)} - \frac{[\xi_1 - \xi + z_0^2(1-\rho(T_1)/\rho(\tau_0))/2]^2}{2z_0^2[1-\rho(T_1)/\rho(\tau_0)]} \right\}. \end{aligned} \quad (37)$$

Comparing Eqs. (31), (37) and Eqs. (7), (8), (14), (15), (23), (24) we can determine an efficiency gain of the maximum likelihood detector in comparison with QL ones and estimate the expediency of its application in each specific case.

VII. CONCLUSION

The obtained results allow making an informed choice of the detector with accounting for the complexity of its hardware implementation and detection efficiency. The simplest algorithm in terms of the hardware implementation is QL algorithm (4). Really, it is only required to form a random variable for the decision making here, and this variable linearly depends on the realization of the observable data. However, QL algorithm (4) is characterized by the lowest detection efficiency. In order to increase the detection efficiency, it is possible to use either QL detector (12) with adaptation on amplitude or QL detector (17) with adaptation on duration. The first of them should generate the random variable which nonlinearly depends on the realization of the observable data, the second detector should search an absolute maximum of the random process which is linear transformation from the realization of the observable data. The maximum likelihood detection algorithm (25) is the most effective and complicated one in the implementation. Indeed, for the decision making it is required to find an absolute maximum of the random process nonlinearly depending on the realization of the observable data. But this complication can be considered as insignificant in many cases.

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REFERENCES

- [1] A.P. Trifonov, "Receiving a signal with unknown duration against white noise", Radio Engineering and Electronic Physics, vol. 22, No. 1, 1977, pp. 90-98.
- [2] A.P. Trifonov, Yu.S. Shinakov, Joint Discrimination of Signals and Estimation of Their Parameters against Background [in Russian]. Radio i Svyaz', Moscow, 1986.
- [3] A.P. Trifonov, Yu.E. Korchagin, "Receiving a signal with unknown duration", Izvestiya Vysshikh Uchebnykh Zavedenij. Radiofizika, vol. 45, No 7, 2002, pp. 625-637.
- [4] A.P. Trifonov, V.K. Buteiko, "Reception of a signal of unknown amplitude and duration on a background of white noise", Izvestiya Vysshikh Uchebnykh Zavedenij. Radioelektronika, vol. 27, No 8, 1981, pp. 28-34.
- [5] V.I. Tikhonov, Optimal Signal Reception [in Russian]. Radio i Svyaz', Moscow, 1983.
- [6] Signal Detection Theory [in Russian], Edited by P.A. Bakut, Radio i Svyaz', Moscow, 1984.
- [7] O.V. Chernoyarov, Sai Si Thu Min, A.V. Salnikova, B.I. Shakhtarin, A.A. Artemenko, "Application of the Local Markov Approximation Method for the Analysis of Information Processes Processing Algorithms with Unknown Discontinuous Parameters", Applied Mathematical Sciences, vol. 8., No. 90, 2014, pp. 4469-4496.
- [8] O.V. Chernoyarov, A.V. Salnikova, A.E. Rozanov, M. Marcokova, "Statistical Characteristics of the Magnitude and Location of the Greatest Maximum of Markov Random Process with Piecewise Constant Drift and Diffusion Coefficients", Applied Mathematical Sciences, vol. 8., No. 147, 2014, pp. 7341-7357.
- [9] E.B. Dynkin, Theory of Markov Processes. Dover Publications Inc., New York, 2006.