

Asymptotic Stability of Runge-Kutta Methods for Linear Delay-integro-differential-Algebraic Equation

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Abstract—This paper deals with the asymptotic behavior of Runge-Kutta methods for linear delay-integro-differential-algebraic equations. It was shown that R-K of A-stability preserves the asymptotic stability of the equations under some conditions if applied to the equations.

Keywords—delay-differential-algebraic equations; Runge-Kutta methods; asymptotic stability

I. THE ASYMPTOTIC STABILITY OF THE EQUATIONS

We consider the delay integral differential algebraic system as follows:

$$\begin{cases} Ax'(t) + Bx(t) + Cx'(t-\tau) + Dx(t-\tau) + G \int_{t-\tau}^t x(s)ds = 0, t \geq 0 \\ x(t) = \phi(t), -\tau \leq t \leq 0 \end{cases} \quad (1)$$

It is assumed that the coefficient matrix $A, B, C, D, G \in R^{d \times d}$ are all upper triangular matrix. $\phi(t) \in C^d$ is a d-dimensional initial vector which is known. The existence and uniqueness of solutions of system (1) is proven in the literature [1][2] as follows:

Theorem 1 If the coefficient matrices of the system (1) meet the following conditions:

For any

$$\gamma \in R^d, \langle \gamma, A\gamma \rangle \geq \langle \gamma, C\gamma \rangle \quad (2)$$

If

$$\det[B + D + \tau G] \neq 0 \quad (3)$$

$$\det[s^2 A + sB + G] = 0, s \neq 0 \Rightarrow \operatorname{Re} s < 0 \quad (4)$$

$$\sup_{\forall \operatorname{Re} s = 0, s \neq 0} \rho[(s^2 A + sB + G)^{-1}(s^2 C + sD - G)] < 1 \quad (5)$$

The system is asymptotically stable^{[3][4]}. (In which $\rho[\cdot]$ indicates the matrix spectral radius.)

II. ASYMPTOTIC STABILITY OF THE RUNGE-KUTTA METHOD

The following Runge-Kutta method

$$y_{n+1} = y_n + \sum_{i=1}^s b_i f(t_n + c_i h, Y_i), \quad (6)$$

$$Y_i = y_n + h \sum_{j=1}^s a_{ij} f(t_n + c_j h, Y_j), i = 1, 2, \dots, s \quad (7)$$

was applied to the scalar test equation:

$$y'(t) = \lambda y, y(t_0) = y_0, \operatorname{Re} \lambda < 0$$

We can get the following conclusion:

$$\begin{aligned} y_{n+1} &= y_n + h\lambda b^T Y \\ Y &= e y_n + h\lambda A Y \end{aligned}$$

In which

$$b = (b_1, b_2, \dots, b_s)^T, Y = (Y_1, Y_2, \dots, Y_s)^T, e = (1, 1, \dots, 1)^T.$$

If the s-th order matrices $(I - h\lambda A)$ is nonsingular, we get

$$y_{n+1} = [1 + h\lambda b^T (I - h\lambda A)^{-1} e] y_n = R(z) y_n$$

in which $z = h\lambda$, $R(z) = 1 + z b^T (I - zA)^{-1} e$, and $R(z)$ is called the stable functions of the method.

Definition 1 If the method (6 7) is applied to the experimental equation, and the stability function satisfies the condition of $|\operatorname{Re}(z)| < 1$, we say that the method is strictly A stable^[5]. $S_R : \{z : \operatorname{Re} z < 0, |R(z)| < 1\}$ is called absolute stability region of the method (6 7).

The s-th order R-K method $(\tilde{A}, \tilde{b}, \tilde{c})$ is applied to the experimental equation (1), in which

$$\begin{aligned} \tilde{A} &= (\tilde{a}_{ij})_{s \times s}, \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_s)^T, 0 < \tilde{c}_i < 1, \text{ and} \\ \tilde{c}_i &= \sum_{j=1}^s \tilde{a}_{ij} \quad (i = 1, 2, \dots, s). \end{aligned}$$

The step is taken as $h = \tau / m$, m is any positive integer, the knot is taken as $t_n = t_0 + nh$

$$AX'_m + B(x_n + \sum_{j=1}^s \tilde{a}_{ij} h X'_{nj}) + CX'_{n-m} + D(x_{n-m} + \sum_{j=1}^s \tilde{a}_{ij} h X'_{nj}) + G \sum_{q=0}^m h v_q(x_{n-q} + \sum_{j=1}^s \tilde{a}_{ij} h X'_{n-qj}) = 0 \quad (8)$$

$$x_{n+1} = x_n + \sum_{i=1}^s \tilde{b}_i h X'_{ni} \quad (9)$$

In which

$$\begin{aligned} X'_{ni} &\approx x'(t_n + c_i h) \\ x_n + \sum_{j=1}^s \tilde{a}_{ij} h X'_{nj} &\approx x(t_n + c_i h) \\ h \sum_{q=0}^m v_q (x_{n-q} + \sum_{j=1}^s \tilde{a}_{ij} h X'_{n-qj}) &\approx \int_{t_n+c}^{t_n+ch} x(s) ds \end{aligned}$$

The coefficient $v_q (q=0,1,\dots,m)$ is determined by the following compound integral formula in which the order of convergence is $p=2s$.

$$\int_0^\tau x(s) ds = h \sum_{q=0}^m v_q x((m-q)h) + O(h^{2s+1})$$

We denote

$$K_{ni} = hX'_{ni} \in C^d, i=1,2,\dots,d$$

$$K_n = [K_{n2}^1, K_{n2}^1, \dots, K_{ns}^1, K_{n1}^2, \dots, K_{ns}^2, \dots, K_{n1}^d, \dots, K_{ns}^d] \in C^{d \times s}$$

So the equations (8)(9) can be written as

$$\begin{aligned} &\begin{bmatrix} A \otimes I_s + h(B + v_0 hG) \otimes \tilde{A} & O \\ -I_d \otimes \tilde{b}^T & I_d \end{bmatrix} \begin{bmatrix} K_n \\ x_{n+1} \end{bmatrix} \\ &+ \begin{bmatrix} v_1 h^2 G \otimes \tilde{A} & h(B + v_0 G) \otimes e \\ O & I_d \end{bmatrix} \begin{bmatrix} K_{n-2} \\ x_{n-1} \end{bmatrix} \\ &+ \begin{bmatrix} v_2 h^2 G \otimes \tilde{A} & h^2 v_1 G \otimes e \\ O & O \end{bmatrix} \begin{bmatrix} K_{n-2} \\ x_{n-1} \end{bmatrix} + \dots + \\ &+ \begin{bmatrix} v_{m-1} h^2 G \otimes \tilde{A} & h^2 v_{m-2} G \otimes e \\ O & O \end{bmatrix} \begin{bmatrix} K_{n-m+1} \\ x_{n-m+2} \end{bmatrix} \\ &+ \begin{bmatrix} C \otimes I_s + h(D + v_m hG) \otimes \tilde{A} & h^2 v_{m-1} G \otimes e \\ O & O \end{bmatrix} \begin{bmatrix} K_{n-m} \\ x_{n-m+1} \end{bmatrix} \\ &+ \begin{bmatrix} O & h(D + v_m hG) \otimes e \\ O & O \end{bmatrix} \begin{bmatrix} K_{n-m-1} \\ x_{n-m} \end{bmatrix} = O \end{aligned}$$

The corresponding characteristic equation is:

$$p(z) = \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \text{ in which:}$$

$$T_{11} = z^{m+1} \left\{ A \otimes I_s + h[B + (v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \dots + \frac{v_{m-1}}{z^{m-1}})hG] \otimes \tilde{A} \right\} + z[C \otimes I_s + h(D + v_m hG) \otimes \tilde{A}]$$

$$T_{12} = z^m \left\{ h[B + (v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \dots + \frac{v_{m-1}}{z^{m-1}})hG] \otimes e \right\} + h(D + v_m hG) \otimes \tilde{A}$$

$$T_{21} = -z^{m+1} I_d \otimes \tilde{b}^T, T_{22} = z^{m+1} I_d - z^m I_d$$

We denote

$$\bar{B} = B + (v_0 + \frac{v_1}{z} + \dots + \frac{v_{m-1}}{z^{m-1}})hG$$

$$\bar{D} = D + v_m hG$$

so we can get that

$$T_{11} = z^{m+1} (A \otimes I_s + h\bar{B} \otimes \tilde{A}) + z(C \otimes I_s + h\bar{D} \otimes \tilde{A})$$

$$T_{12} = z^m h\bar{B} \otimes e + h\bar{D} \otimes \tilde{A}$$

By the difference equation theory we know that If the characteristic equation satisfies the condition:

$$p(z) = 0 \Rightarrow |z| < 1$$

we have $x_n \rightarrow 0 (n \rightarrow \infty)$, That is, the numerical method is asymptotically stable^[5]. So we can Prove $p(z) \neq 0$ under certain conditions $|z| \geq 1$ as follows:

First we prove T_{11} is a nonsingular matrix. By assuming we know \bar{B}, \bar{D} are also upper triangular matrices, its main diagonal elements are set respectively as $\bar{b}_i, \bar{d}_i (i=1,2,\dots,d)$, then

$$\begin{aligned} \det T_{11} &= \prod_{i=1}^d \det [z^{m+1} (a_i I_s + h\bar{b}_i \tilde{A}) + z(c_i I_s + h\bar{d}_i \tilde{A})] \\ &= \prod_{i=1}^d \det [z^{m+1} (a_i + c_i z^{-m}) I_s + h z^{m+1} (\bar{b}_i + \bar{d}_i z^{-m}) \tilde{A}] \\ &= \prod_{i=1}^d q_i(z) \end{aligned}$$

If $a_i \neq 0$, We suppose that $c_i \neq 0$, $|a_i| > |c_i|$ and $|z| \geq 1$, we can conclude that $\text{Re} \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} > 0$,

So $a_i + c_i z^{-m} \neq 0, (|z| \geq 1)$ and

$$q_i(z) = \det [z^{m+1} (a_i + c_i z^{-m}) (I_s + h \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} \tilde{A})]$$

For all the eigenvalues of \tilde{A} are real, we can get $q_i(z) \neq 0$; and some conclusions:

1. If $a_i = 0$, we suppose $c_i = 0$, if $|z| \geq 1$ and $\bar{b}_i + \bar{d}_i z^{-m} \neq 0$, we can conclude that T_{11} is nonsingular:

$$q_i(z) = \det [z^{m+1} h(\bar{b}_i + \bar{d}_i z^{-m}) \tilde{A}] \neq 0$$

So we have

$$\begin{aligned} p(z) &= \det \begin{bmatrix} T_{11} & T_{12} \\ O & T_{22} - T_{21} T_{11}^{-1} T_{12} \end{bmatrix} = \det[T_{11}] \cdot \det[T_{22} - T_{21} T_{11}^{-1} T_{12}] \\ &= \prod_{i=1}^d z^m \left\{ z - (1 - \tilde{b}^T [(a_i + c_i z^{-m}) I_s + h(\bar{b}_i + \bar{d}_i z^{-m}) \tilde{A}]^{-1} h(\bar{b}_i + \bar{d}_i z^{-m}) e) \right\} \end{aligned}$$

For

$1 - \tilde{b}^T [(a_i + c_i z^{-m})I_s + h(\bar{b}_i + \bar{d}_i z^{-m})\tilde{A}]^{-1} h(\bar{b}_i + \bar{d}_i z^{-m})e$
 $= 1 - \tilde{b}^T \tilde{A}^{-1} e$, by the strict stability of Runge-Kutta method,
we can get that: $|1 - \tilde{b}^T \tilde{A}^{-1} e| < 1$.

And on the condition of $|z| \geq 1$ and

$$z^m \left\{ z - (1 - \tilde{b}^T [(a_i + c_i z^{-m})I_s + h(\bar{b}_i + \bar{d}_i z^{-m})\tilde{A}]^{-1} h(\bar{b}_i + \bar{d}_i z^{-m})e) \right\} \neq 0$$

we also can get that $\det[T_{11}] \det[T_{22} - T_{21} T_{11}^{-1} T_{12}] \neq 0$,

and $p(z) \neq 0, (|z| \geq 1)$

2. If $a_i \neq 0$, $c_i \neq 0$, We can get
 $a_i + c_i z^{-m} \neq 0, (|z| \geq 1)$ and

$$1 - \tilde{b}^T [(a_i + c_i z^{-m})I_s + h(\bar{b}_i + \bar{d}_i z^{-m})\tilde{A}]^{-1} h(\bar{b}_i + \bar{d}_i z^{-m})e = 1 - R(-h \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}})$$

$$= \tilde{b}^T (I_s + h \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} \tilde{A})^{-1} h \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} e$$

In which $R(\hat{z}) = 1 - \hat{z} \tilde{b}^T (I_s - \hat{z} \tilde{A})^{-1} e$ is the stable functions of the strictly A stable Runge-Kutta method.

If $-h \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} \in S_R$ is satisfied, in which S_R is stability range of the method, we can get that: on the condition of $|z| \geq 1$, the formula

$$z^m \left\{ z - (1 - \tilde{b}^T [(a_i + c_i z^{-m})I_s + h(\bar{b}_i + \bar{d}_i z^{-m})\tilde{A}]^{-1} h(\bar{b}_i + \bar{d}_i z^{-m})e) \right\} \neq 0$$

is established.

III. CONCLUSION

Theorem 2 If the coefficient matrix of the system (1) and the chosen parameter $v_q (q = 0, 1, \dots, m)$ of the integral formula satisfy:

(1) For any $\gamma \in R^d$, $|\langle \gamma, A\gamma \rangle| \geq |\langle \gamma, C\gamma \rangle|$ is established.

(2) If $a_i = 0$, we have $c_i = 0 (i = 1, 2, \dots, d)$.

(3) If $a_i \neq 0$, we have $c_i \neq 0$ and $|a_i| > |c_i|$,

$(i = 1, 2, \dots, d)$

(4) If $|z| \geq 1$, and $\text{Re} \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} > 0$

in which

$$\bar{b}_i = b_i + (v_0 + \frac{v_1}{z} + \dots + \frac{v_{m-1}}{z^{m-1}}) h g_i$$

$$\bar{d}_i = d_i + v_m h g_i, \quad i = 1, 2, \dots, d$$

We get the conclusion: If $-h \frac{\bar{b}_i + \bar{d}_i z^{-m}}{a_i + c_i z^{-m}} \in S_R$ is satisfied,

in which S_R is the stability range of the method, The numerical solution of the A strict stable Runge-Kutta method $(\tilde{A}, \tilde{b}, c)$ is asymptotically stable, in which \tilde{A} satisfies that the characteristic values of \tilde{A} are all real.

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