Stability of Balanced Implicit Method for Hybrid Stochastic Differential Equation

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***Abstract*—The paper shows that the balanced implicit method reproduces the moment exponentially stable for hybrid stochastic differential equation. The balanced implicit method has better behavior of stability in comparison with the explicit Euler-Maruyama method. Experiments simulation indicates that the balanced implicit method may overcome unstability of the Euler-Maruyama method.**

Keywords-stochastic differential equation, balanced implicit method, markovian switching, moment exponentially stability

I. INTRODUCTION

In the 1970 s, Black and Scholes [1] proposed:

$$\mathrm{d}x(t) = \mu x(t)\,\mathrm{d}t + \sigma x(t)\mathrm{d}\omega(t), \tag{1}$$

and he got the Nobel Prize in economics in 1993 for this model. In this model, the terminology and notation are defined as follows:

$\omega(t)$: A standard Winner Process, μ : Asset returns

σ : Fluctuation ratio

In the classical Black-Scholes model, μ and σ are constants. While Hull and White [2, 3], Yin and Zhou[4] found μ and σ are Markov processes instead of constants. So the Markov switching system is proposed:

$$\mathrm{d}x(t) = \mu(r(t))\,x(\mathrm{t})\,\mathrm{d}t + \sigma(r(t))\,x(t)\mathrm{d}\omega(t), \tag{2}$$

In this model, $r(t)(t>0)$ is a Markov Chain, and its value ranges is a finite state space $S=\{1,2,\ldots,N\}$ .The Markov switching system is popularly used in many financial models, such as in pricing bonds, mortgage bonds, convertible bonds etc.. [5]. However, most of the stochastic differential equation cannot get explicit solution, the numerical method is commonly concerned, various numerical methods are put forward, such as Euler's method, the implicit Euler method, the backward Euler method etc.. [10]. Implicit Euler method with its better stability, gets more attention, especially the backward Euler method which is often used in the nonlinear systems. But implicit Euler method is only applicable to rigid equation with little noise because this method only partly uses the implicit item in drift coefficient. In this paper, by means of discrete Ito Lemma, we demonstrate that the balanced implicit method torque index of markov switching asset pricing model is stable, and at the same time we give the instability condition and conclude that balanced implicit method is more stable than Euler's method with numerical method.

II. BALANCED IMPLICIT METHOD

Let $|x|, x\in R^n$ be a Euclidean norm, If A is a vector or matrix，its transpose is denoted by A^{T}, and if A is a matrix, its trace norm is denoted by $|A|=\sqrt{\mathrm{trace}\left(A^{\mathrm{T}}A\right)}$,while its operator norm is denoted by $\|A\|=\sup\{|Ax|:|x|=1\}$ .We let $\left(\Omega, F,(F_t)_{t\geq 0}, P\right)$ be a complete probability space with a filtration $(F_t)_{t\geq 0}$,satisfying the usual conditions (i.e. incremental, right of continuous, F_0 Contains all the zero measure set); $\omega(t)$ is a standard Brown motion in probability space; $r(t)(t\geq 0)$ is a right continuous markov chain whose value range is the finite state space $S=\{1,2,\ldots,N\}$,it generates $\Gamma=(\gamma_{\mathrm{ij}})_{N\times N}$ and satisfies

$$P\{r(t+\Delta)=j|r(t)=i\}=\begin{cases}\gamma_{ij}\Delta+0(\Delta), & if\ \ i\neq j,\\ 1+\gamma_{ii}\Delta+0(\Delta), & if\ \ i=j,\end{cases}$$

where $\Delta>0, \gamma_{ij}\geq 0$ is the transition probability from i to j ,and when $i\neq j$, $\gamma_{ii}=-\sum_{j\neq i}\gamma_{ij}$.

Considering that $r(t)$ is a right successive step function with limited multiple jump discontinuity point in R_+ , we assume that the markov chain $r(t)$ is independent of Brownian movement $\omega(t)$. We also assume that the markov chain is irreducible, that is for any $i,j\in S,$ exists

$i_1, i_2, \cdots, i_k \in S$, which can make sure $\gamma_{i,i_1}\gamma_{i_1,i_2}\cdots\gamma_{i_k,j} > 0$ and 0 is an Eigen value root of Γ, and $\mathrm{rank}(\Gamma)=N\text{-}1$.So there is a unique stationary distribution $\pi=(\pi_1,\ \pi_2,\cdots,\ \pi_N)\in R^{1\times N}$, which satisfies $\pi\Gamma=0, \sum_{j=1}^{N}\pi_j = 1, \pi_j > 0,\ j\in S.$

Lemma 2.1([6]) We assume that for any $\Delta > 0, k \ge 0, r_k^{\Delta} = r(k\Delta)$, then $\{r_k^{\Delta},\ \ k = 0,1,2\ldots\}$ is a discrete markov chain, whose One-step transition probability matrix array is $P(\Delta)=(P_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}$.

Given the Linear markov switching stochastic differential equation:

$$\mathrm{d}x(t) = \mu(r(t))x(t)\mathrm{d}t + \sigma(r(t))x(t)\mathrm{d}\omega(t), \tag{3}$$

on $t \ge 0, x(0) = x_0 \in R, r(0) = r_0 \in S,$ and r_0 is a measurable random variables of S-Value F_t.

Since γ_{ij} is independent of x, and the path of r is produced before x and independent of x. We define $\Delta \in (0,1)$ as step size, then we can get discrete markov chain $\{r_k^{\Delta},\ \ k = 0,1,2,\ldots\}$ as follows: calculate the one-step transition probability matrix $P(\Delta)$ .Let $r_0^{\Delta}=i_0$ and get a random number ξ_1 which distributed evenly over [0,1].Then we define

$$r_1^{\Delta}=\begin{cases} i_1 & i_1 \in S-\{N\}, \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \le \xi_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\ N, & \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \le \xi_1, \end{cases}$$

on $\sum_{j=1}^{0} P_{i_0,j}(\Delta)=0.$ then get another random number ξ_2 which is distributed evenly over [0,1] and is independent of ξ_1. We define

$$r_2^{\Delta}=\begin{cases} i_2 & i_2 \in S-\{N\}, \sum_{j=1}^{i_2-1} P_{r_1^{\Delta},j}(\Delta) \le \xi_2 < \sum_{j=1}^{i_2} P_{r_1^{\Delta},j}(\Delta), \\ N, & \sum_{j=1}^{N-1} P_{r_1^{\Delta},j}(\Delta) \le \xi_2. \end{cases}$$

Repeating this process, we can get $\{r_k^{\Delta},\ \ k = 1,2,\ldots\}$.This process can independently produce more trajectory, after we get the discrete markov chain $\{r_k^{\Delta},\ \ k = 1,2,\ldots\}$, we define the balanced implicit method of hybrid system (3).For $t_k = k\Delta, k > 0.$ $X_0=x(t_0), r_0^{\Delta} = i_0,$ here is the approximate value for $X_k \approx x(t_k)$:

$$X_{k+1} = X_k + \mu(r_k^{\Delta})X_k\Delta + \sigma(r_k^{\Delta})X_k\Delta\omega_k + C(X_k)(X_k - X_{k+1}), \tag{4}$$

where $\Delta\omega_k=\omega(t_{k+1})-\omega(t_k)$ is a F_{t_k}-measurable random variable which is independent and Obey the $N(0,\ \Delta)$ distribution. $C(X_k)$ satisfies $C(X_k)=c_{0k}\Delta + c_{1k}|\Delta\omega_k|,$ $c_{0k}=C_0(X_k),\ c_{1k}=C_1(X_k)$ is known as control function, and often is chosen as constant[7].

Hypothesis 2.2: Assume that c_{0k} and c_{1k} are limited function, for any step size Δ, $\overline{\alpha}_0 \ge \Delta,$ $\alpha_0 \in [0,\overline{\alpha}_0]$, $\alpha_1 \ge 0,$ $M(x)=1+\alpha_0 c_{0k}(x)+\alpha_1 c_{1k}(x)$ and $\left|(M(x))^{-1}\right| \le C < \infty, C > 0.$ The equilibrium methods (4)is very general, it covers the implicit and semi implicit Euler method. If denoted by $C(X_k)=(\theta\text{-}1)\mu(r_k^{\Delta})\Delta,$ and from (4), we can get the following semi implicit method

$$X_{k+1} = X_k + \theta\mu(r_k^{\Delta})X_k\Delta + (1-\theta)\mu(r_k^{\Delta})X_{k+1}\Delta + \sigma(r_k^{\Delta})X_k\Delta\omega_k,$$

then for $\theta = 1,$ we can get the Euler's method

$$X_{k+1} = X_k + \mu(r_k^{\Delta})X_k\Delta + \sigma(r_k^{\Delta})X_k\Delta\omega_k,$$

Next we assume that c_{0k}, c_{1k} is independent of the constant k ,and $c_0=c_{0k},\ \ c_1=c_{1k}.$

It is easy to prove that balanced implicit method and Euler method have strong convergence with the same 1/2 order. This article will take all the constants and all the product with constants as C except c_0, c_1,

III. STABILITY

Lemma 3.1([6]) we assume that μ is a random variables of standard normal distribution, $\phi(x) \in C^3(R), |\phi'''(x)| \le L,$ and L is a constant $x \in [1\text{-}\delta, 1+\delta](0 < \delta < 1), \phi : R \to R$ is a Lebesgue integrable function, then for $c_1, c_2, c_3 \in R,\ \ \Delta \to 0,$

$$E\left[\phi\left(1+c_1\Delta + c_2\sqrt{\Delta}\mu + c_3\sqrt{\Delta}|\mu|\right)\right] = \phi(1) +$$
$$\phi'(1)\left(c_1\Delta + \frac{2}{\sqrt{2\pi}}c_3\sqrt{\Delta}\right) + \frac{\phi''(1)}{2}\left(c_2^2 + c_3^2\right)\Delta + o(\Delta).$$

Theorem 3.2 Assume that there is a positive constant Δ^*, for any $\Delta < \Delta^*$, from the balanced implicit method (4) we can get

$$\lim_{n\to\infty}\sup \frac{1}{n\Delta}\log E\left|X_n\right|^p \le \frac{p}{\Delta}\log C + \sum_{j\in S}\pi_j p\left[\begin{array}{l}\frac{4}{\sqrt{2\pi}}c_1\Delta^{-\frac{1}{2}}+\mu_j+c_0\\ +\frac{p-1}{2}\left(\sigma_j^2+c_1^2\right)+C\Delta^{\frac{1}{2}}+o\left(\Delta^{\frac{1}{2}}\right)\end{array}\right].$$

that is, for a sufficiently small step size Δ, if

$$\frac{p}{\Delta}\log C + \sum_{j\in S}\pi_j p\left[\begin{array}{l}\frac{4}{\sqrt{2\pi}}c_1\Delta^{-\frac{1}{2}}+\mu_j+c_0\\ +\frac{p-1}{2}\left(\sigma_j^2+c_1^2\right)+C\Delta^{\frac{1}{2}}+o\left(\Delta^{\frac{1}{2}}\right)\end{array}\right]<0 \tag{5}$$

then the balanced implicit method(4)is p -th moment exponential stable.

Proof From the sequence(4), we can get

$$E\left[\left|X_{n+1}\right|^p\right] = E\left(\left|X_0\right|^p\prod_{k=0}^{n-1}Z_k^p E\left[Z_n^p\middle|F_{t_n}\right]\right), \tag{6}$$

$F_{t_n}=\sigma\left(\left\{r(u)\right\}_{u\ge 0,}\left\{\omega(s)\right\}_{0\le s\le t_n}\right)$ is a σ -algebra, produced by $\left\{r(u)\right\}_{u\ge 0,}$ and $\left\{\omega(s)\right\}_{0\le s\le t_n}$,through the assumption, calculate that

$$\begin{aligned}Z_n^p &= \left|\frac{1+c_0\Delta+c_1\left|\Delta\omega_n\right|+\mu\left(r_n^\Delta\right)\Delta+\sigma\left(r_n^\Delta\right)\Delta\omega_n}{1+c_0\Delta+c_1\left|\Delta\omega_n\right|}\right|^p\\ &\le C^p\left\{\begin{array}{l}1+2\left[\left(c_0+\mu\left(r_n^\Delta\right)\right)\Delta+c_1\left|\Delta\omega_n\right|+\sigma\left(r_n^\Delta\right)\Delta\omega_n\right]\\ +\left[\left(c_0+\mu\left(\gamma_n^\Delta\right)\right)\Delta+c_1\left|\Delta\omega_n\right|+\sigma\left(r_n^\Delta\right)\Delta\omega_n\right]^2\end{array}\right\}^{\frac{p}{2}}.\end{aligned} \tag{7}$$

and from the Taylor expansion, let

$$\begin{aligned}u=&2\left[\left(c_0+\mu\left(r_n^\Delta\right)\right)\Delta+c_1\left|\Delta\omega_n\right|+\sigma\left(r_k^\Delta\right)\Delta\omega_n\right]\\ &+\left[\left(c_0+\mu\left(r_n^\Delta\right)\right)\Delta+c_1\left|\Delta\omega_k\right|+\sigma\left(r_n^\Delta\right)\Delta\omega_n\right]^2\end{aligned}$$

and choose $\Delta, c_0, c_1,$ which can make sure that $u>-1,$ then

$$E\left[Z_n^p\middle|F_{t_n}\right]\le C^P E\left[\begin{array}{l}1+\frac{p}{2}u+\frac{p(p-2)}{8}u^2\\ +\frac{p(p-2)(p-4)}{48}u^3\end{array}\middle|F_{t_n}\right]. \tag{8}$$

Let $u_1=\left(c_0+\mu\left(r_n^\Delta\right)\right)\Delta+c_1\left|\Delta\omega_n\right|+\sigma\left(r_k^\Delta\right)\Delta\omega_n,$ then $u=2u_1+u_1^2$. It is easy to calculate that

$$\begin{aligned}&E\left[\left(\Delta\omega_n\right)^{2i}\right]=(2i-1)!!\Delta^i, E\left[\left(\Delta\omega_n\right)^{2i-1}\right]=0,\\ &E\left[\left|\Delta\omega_n\right|^{2i-1}\right]=\frac{2^i}{\sqrt{2\pi}}(i-1)!\Delta^{\frac{2i-1}{2}},\end{aligned} \tag{9}$$

on $\begin{array}{l}(2i-1)!!=(2i-1)(2i-3)\cdots 3\cdot 1,\\ (i-1)!=(i-1)(i-2)\cdots 2\cdot 1, i=1,2\cdots.\end{array}$ so

$$E\left[\mu_1\middle|F_{t_n}\right]=\frac{2}{\sqrt{2\pi}}c_1\Delta^{\frac{1}{2}}+\left(c_0+\mu\left(r_n^\Delta\right)\right)\Delta, \tag{10}$$

$$\begin{aligned}E\left[\mu_1^2\middle|F_{t_n}\right]=&\left(c_1^2+\sigma^2\left(r_n^\Delta\right)\right)\Delta+\frac{4}{\sqrt{2\pi}}c_1\left(c_0+\mu\left(r_n^\Delta\right)\right)\Delta^{\frac{3}{2}}\\ &+\left(c_0+\mu\left(r_n^\Delta\right)\right)^2\Delta^2,\end{aligned} \tag{11}$$

$$E\left[\mu_1^3\middle|F_{t_n}\right]=C\Delta^{\frac{3}{2}}+o\left(\Delta^{\frac{3}{2}}\right), E\left[\mu_1^4\middle|F_{t_n}\right]=o\left(\Delta^{\frac{3}{2}}\right). \tag{12}$$

plug (10)-(12) into (5), then

$$E\left[\left|X_{n+1}\right|^p\right]\le E\left(\left|X_0\right|^p C^p\left(\begin{array}{l}1+\frac{2}{\sqrt{2\pi}}pc_1\Delta^{\frac{1}{2}}\\ +p\left[\begin{array}{l}\mu\left(r_n^\Delta\right)+c_0\\ +\frac{p-1}{2}\left(\sigma^2\left(r_n^\Delta\right)\right)\\ +c_1^2\end{array}\right]\Delta\\ +C\Delta^{\frac{3}{2}}+o\left(\Delta^{\frac{3}{2}}\right)\end{array}\right)\prod_{k=0}^{n-2}Z_k E\left[Z_{n-1}\middle|F_{t_{n-1}}\right]\right) \tag{13}$$

Repeat this process, then

$$E\left[\left|X_{n+1}\right|^p\right]\le E\left(\left|X_0\right|^p\exp\left[\begin{array}{l}p(\mathrm{n}+1)\log C\\ +\sum_{k=0}^{n}\log\left(\begin{array}{l}1+\frac{2}{\sqrt{2\pi}}pc_1\Delta^{\frac{1}{2}}\\ +p\left[\begin{array}{l}\mu\left(r_k^\Delta\right)+c_0\\ +\frac{p-1}{2}\left(\sigma^2\left(r_k^\Delta\right)+c_1^2\right)\end{array}\right]\Delta\\ +C\Delta^{\frac{3}{2}}\end{array}\right)+o\left(\Delta^{\frac{3}{2}}\right)\end{array}\right]\right).$$

By markov chain ergodicity and the inequation $\log(1+x)\le x,$ $\forall x>-1$ we can figure out

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\log\left(1+\frac{2}{\sqrt{2\pi}}pc_1\Delta^{-\frac{1}{2}}+p\left[\begin{array}{l}\mu\left(r_k^{\Delta}\right)+c_0\\+\frac{p-1}{2}\left(\sigma^2\left(r_k^{\Delta}\right)+c_1^2\right)\end{array}\right]\Delta+C\Delta^{\frac{3}{2}}+o\left(\Delta^{\frac{3}{2}}\right)\right)$$

$$\le\sum_{j\in S}\pi_j p\left[\frac{2}{\sqrt{2\pi}}c_1\Delta^{-\frac{1}{2}}+\left[\mu_j+\frac{p-1}{2}\sigma_j^2\right]\Delta+\left(c_0+\frac{p-1}{2}c_1^2\right)\Delta+C\Delta^{\frac{3}{2}}+o\left(\Delta^{\frac{3}{2}}\right)\right]a.s.$$

Denoted by

$$\lambda=\frac{p}{\Delta}\log C+\sum_{j\in S}\pi_j p\left[\begin{array}{l}\frac{4}{\sqrt{2\pi}}c_1\Delta^{-\frac{1}{2}}+\mu_j+c_0\\+\frac{p-1}{2}\left(\sigma_j^2+c_1^2\right)+C\Delta^{\frac{1}{2}}+o\left(\Delta^{\frac{1}{2}}\right)\end{array}\right].$$

then there exists $\varepsilon>0$, which make

$$\lim_{n\to\infty}\frac{1}{n+1}\left[\begin{array}{l}p(n+1)\log C\\+\sum_{k=0}^{n}\log\left(\begin{array}{l}1+\frac{2}{\sqrt{2\pi}}pc_1\Delta^{-\frac{1}{2}}+p\left[\begin{array}{l}\mu\left(r_k^{\Delta}\right)+c_0\\+\frac{p-1}{2}\left(\sigma^2\left(r_k^{\Delta}\right)+c_1^2\right)\end{array}\right]\Delta\\+C\Delta^{\frac{3}{2}}+o\left(\Delta^{\frac{3}{2}}\right)\end{array}\right)\end{array}\right] \quad (14)$$

$<\lambda+\varepsilon$ *a.s.*

According to the lemma Fatou,

$$\lim_{n\to\infty}e^{-(\lambda+\varepsilon)(n+1)\Delta}E\left|X_{n+1}\right|^p=0$$

If choose $c_1<0, c_0+\frac{p-1}{2}c_1^2<0, C<1,$ then criteria(5) degenerates to $\sum_{j\in S}\pi_j p\left(\mu_j+\frac{p-1}{2}\sigma_j^2\right)<0,$

This is the sufficient condition for p -th moment exponential stability of Euler's method.

Theorem 3.2 indicates that even $\sum_{j\in S}\pi_j p\left(\mu_j+\frac{p-1}{2}\sigma_j^2\right)>0$, we also can make criteria (5) possible when the controls parameters c_1, c_0 and Δ are chosen properly. Combined with the Euler's method., balanced implicit method is more stable.

Next is the sufficient condition for instability of method (4)

Theorem 3.3 Balanced implicit method (4)satisfies

$\lim_{n\to\infty}\frac{1}{n\Delta}\log\left(E\left|X_n\right|^p\right)\ge\sum_{j\in S}p\pi_j\left(\mu_j-\frac{1}{2}\sigma_j^2\right)$. That is if $\sum_{j\in S}\pi_j\left(\mu_j-\frac{1}{2}\sigma_j^2\right)>0$, for any sufficiently small step size Δ, balanced implicit method is torque index is not stable.

Proof

from (4), we can get

$$\left|X_{n+1}\right|^p=\left|X_0\right|^p\prod_{k=0}^{n}Z_k^p=\left|X_0\right|^p\exp\left(\sum_{k=1}^{n}\log Z_k^p\right).$$

$$E\left[\left|X_{n+1}\right|^p\right]=E\left(\begin{array}{l}\left|X_0\right|^p\exp\left(\sum_{k=1}^{n-1}\log Z_k^p\right)\\E\left[\exp\left(\log Z_k^p\right)\middle|F_{t_n}\right]\end{array}\right). \quad (15)$$

Depend on the Jensen inequation, we can get

$$E\left[\exp\left(\log Z_n^p\right)\middle|F_{t_n}\right]\ge\exp\left(E\left[\log Z_n^p\middle|F_{t_n}\right]\right). \quad (16)$$

use the lemma 3.1,and combine (23),we can get

$$E\left[\left|X_{n+1}\right|^p\right]\ge E\left(\begin{array}{l}\left|X_n\right|^p\exp\left[\begin{array}{l}p\left(\mu\left(r_k^{\Delta}\right)-\frac{1}{2}\sigma^2\left(r_k^{\Delta}\right)\right)\Delta\\+o(\Delta)\end{array}\right]\exp\left(\sum_{k=1}^{n-2}\log Z_k^p\right)\\E\left[\exp\left(\log Z_{n-1}^p\right)\middle|F_{t_{n-1}}\right]\end{array}\right). \quad (17)$$

Repeat this process, we find

$$E\left[\left|X_{n+1}\right|^p\right]\ge E\left[\left|X_0\right|^p\exp\left(\sum_{k=0}^{n}\left[p\left(\begin{array}{l}\mu\left(r_k^{\Delta}\right)\\-\frac{1}{2}\sigma^2\left(r_k^{\Delta}\right)\end{array}\right)\Delta+o(\Delta)\right]\right)\right].$$

Given (17), when $\varepsilon>0,$ we can get

$$e^{-(\lambda-\varepsilon)n\Delta}E\left|X_n\right|^p\ge E\left|X_0\right|^p\exp\left[\begin{array}{l}-(\lambda-\varepsilon)n\Delta\\+\sum_{k=0}^{n}p\left(\mu\left(r_k^{\Delta}\right)-\frac{1}{2}\sigma^2\left(r_k^{\Delta}\right)\right)\Delta+o(\Delta)\end{array}\right].$$

Depend on the ergodicity of Markov chain, we can get

$$\lim_{n\to\infty}\frac{1}{n\Delta}\sum_{k=1}^{n}\left[p\left(\mu\left(r_k^{\Delta}\right)-\frac{1}{2}\sigma^2\left(r_k^{\Delta}\right)\right)\Delta+o(\Delta)\right]=\sum_{j\in S}\pi_j\left[p\left(\mu_j-\frac{1}{2}\sigma_j^2\right)+o(\Delta)\right].$$

Let $\lambda=\sum_{j\in S}\pi_j p\left(\mu_j-\frac{1}{2}\sigma_j^2\right),$ then

$$\lim_{n\to\infty}\left[\begin{array}{l}-p(\lambda-\varepsilon)n\Delta+\sum_{k=1}^{n}p\left(\mu\left(r_k^{\Delta}\right)-\frac{1}{2}\sigma^2\left(r_k^{\Delta}\right)\right)\Delta\\+o(\Delta)\end{array}\right]=+\infty\ a.s..$$

$$\lim_{n\to\infty}\inf\frac{1}{n\Delta}\log E\left|X_n\right|^p\ge\sum_{j\in S}\pi_j p\left(\mu_j-\frac{1}{2}\sigma_j^2\right)-\varepsilon.$$

so

when $\varepsilon\to 0,$ then

$$\lim_{n\to\infty}\inf\frac{1}{n\Delta}\log E\left|X_n\right|^p\ge\sum_{j\in S}\pi_j p\left(\mu_j-\frac{1}{2}\sigma_j^2\right).$$

Theorem 3.3 indicates that the condition of instability for method (4) is stronger than Euler's, that's balanced implicit method has better stability than Euler's method.

IV. NUMERICAL EXPERIMENT

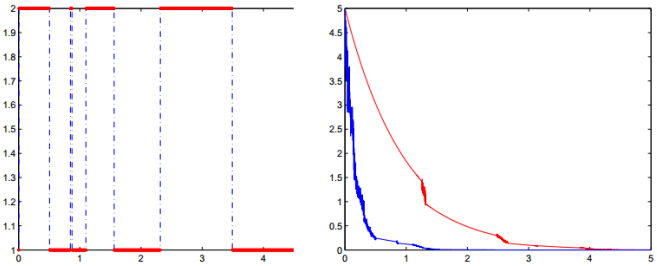

FIGURE I. LEFT: THE PATH OF $r(t)$; RIGHT: THE PATH OF EULER'S METHOD(RED LINE)AND BALANCED IMPLICIT METHOD (BLUE LINE) WHEN THE STEP SIZE $\Delta=0.001$

In this section we will verify the correctness of the conclusion, through the Numerical experiment we confirmed that the balanced implicit method has better stability. Consider a scalar equation

$$\mathrm{d}x(t) = \mu(r(t))x(t)\,\mathrm{d}t + \sigma(r(t))x(t)\mathrm{d}\omega(t), \tag{18}$$

and in it $t \geq 0, \omega(t)$ is scalar Brownian motion, $r(t)$ is a right successive markov chain which values in $S=\{1,2\}$,and generates $\Gamma=\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$.

Assume that $r(t)$ is independent of $\omega(t)$ .It is easy to know $\pi_1=\frac{2}{3},\ \pi_2=\frac{1}{3}$. and when $(x(0),\ r(0))=(5,1)$, the horizontal axis represents time variable t .In the figure 1,2 and 4, Red line represents the curve orbit of Euler's method, the blue line represents the orbit curve of balanced implicit method. In figure 3, red line represents the curve orbit of balanced implicit method when $c_0=-1,\ c_1=-2$ while the blue line represents the orbit curve of balanced implicit method when $c_0=-0.1,\ c_1=-0.2$.

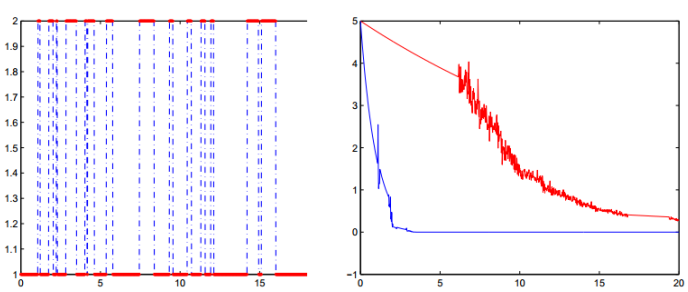

FIGURE II. LEFT: THE PATH OF $r(t)$: RIGHT ; THE PATH OF EULER'S METHOD(RED LINE)AND BALANCED IMPLICIT METHOD (BLUE LINE) WHEN THE STEP SIZE $\Delta=0.001$

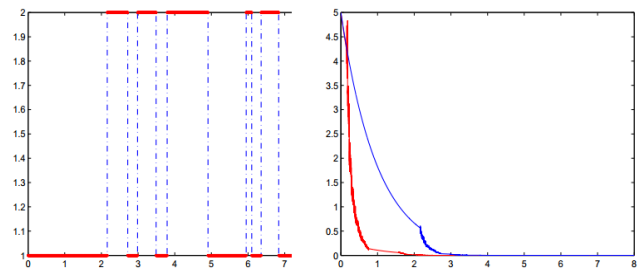

FIGURE III. LEFT: THE PATH OF $r(t)$; RIGHT: WHEN THE STEP SIZE IS $\Delta=0.001, T=8.$ THE RED LINE REPRESENTS THE PATH OF BALANCED IMPLICIT METHOD WHEN $c_{0n}=-1, c_{1n}=-2$,AND BLUE LINE REPRESENTS THE PATH OF BALANCED IMPLICIT METHOD WHEN $c_{0n}=-0.1, c_{1n}=-0.2$

Case 1 $\mu(1)=-1, \sigma(1)=0, \sigma(2)=2$

In this condition, we can find $\sum_{j\in S}\pi_j p\left(\mu_j-\frac{1}{2}\sigma_j^2\right)=-\frac{4}{3}$, which indicate that the equation(27)is almost asymptotic stable everywhere, figure 1 include the Orbit curve for both Euler's method(when $\Delta=0.001,\ T=5$, $c_0=0,\ c_1=0$)and the balanced implicit method (when $\Delta=0.001,\ T=5$, $c_0=-0.1,\ c_1=-2$);figure 2 include the Orbit curves for both Euler's method and the balanced implicit method when $h=0.001,\ T=20,\ c_{0n}=-0.1, c_1=-0.2$ .These two figures indicate that balanced implicit method has better stability than Euler's method. In figure 3, we can find the Orbit curves for balanced implicit method when $h=0.001,\ T=8$ and control coefficient respectively are $c_0=-1,\ c_1=-2$ and $c_0=-0.1,\ c_1=-0.2$, which indicate that the stability can be influenced by the control coefficients c_0 and c_1.

Case 2 $\mu(1)=1.2, \sigma(1)=1, \mu(2)=1.2, \sigma(2)=1.$

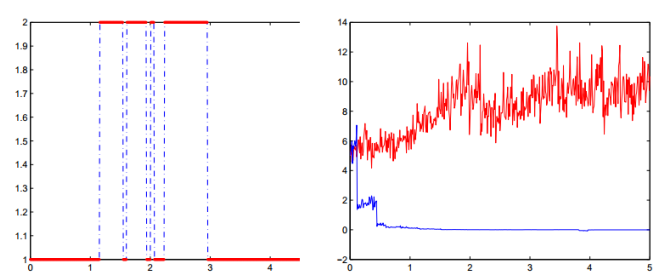

FIGURE IV. LEFT: THE PATH OF $r(t)$; RIGHT: THE CURVE OF EULER'S METHOD(RED LINE)AND BALANCED IMPLICIT METHOD (BLUE LINE) WHEN $h=0.001,\ T=5$, $c_0=-1,\ c_1=-2$

We can figure out that $\sum_{j\in S}\pi_j p\left(\mu_j-\frac{1}{2}\sigma_j^2\right)>0$, and Euler's method is not stable. Figure 4 shows orbital curve of balanced implicit method and Euler's method when h=0.001, T=5, c_0=-1, c_1=-2 ,we can draw a conclusion from the figure that balanced implicit method is stable, while Euler's method is not, that is balanced implicit method can keep stable by choosing appropriate control coefficient, this further illustrates that the balanced implicit method can overcome the instability of Euler method, and makes the system stable.

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