Some Connectedness and Related Property of Hyperspace with Vietoris Topology

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Abstract—For a Hausdorff space $X$, we denote by $2^X$ the collection of all closed subsets of $X$. In this paper, we discuss the connectedness and locally connectedness of hyperspace $2^X$ endowed with the Vietoris topology. Further path connectedness is investigated. The results generalize some theorems of E. Michael.

Keywords—connectedness; locally connectedness; path connectedness; Vietoris topology; hyperspace

I. INTRODUCTION

There are many different compatible topologies on hyperspace $2^X$. Among these topologies, it is well known that finite topology is an important topology. It is called Vietoris topology.

In 1951, E. Michael [1] made a systematic discussion on hyperspace properties with the fine topology. In this paper, the connectedness and related properties of hyperspace $2^X$ with Vietoris topology are discussed. The results improve some theorems of E. Michael.

Definition 1.1 Let $X$ be topology space. By $2^X$ we denote the family of nonempty closed subset of $X$, and then $\{<U>|U \in T\} \cup \{<X,V>|V \in T\}$ is a sub base to a topology $T_V$ in $2^X$.

$T_V$ is called the finite topology in $2^X$ or Vietoris topology.

Obviously, $\{<U_1,U_2,\ldots,U_n>|U_i \in T, i \leq n, n \in N\}$ is a base of Vietoris topology, where

$\langle U_1, U_2, \ldots, U_n \rangle := \{E \in 2^X | E \in \bigcup_{i=1}^n U_i, E \cap U_i \neq \emptyset, \forall i \leq n\}$

$\mathcal{Z}(X) = \{E \in 2^X | E \subset X, E \text{ is a nonempty compact in } X\}$.

For simplicity, we denote by

$n(X) = \{E \in 2^X | E \text{ has } n \text{ elements in } X \text{ at most}\}$;

$(X) = \{E \in 2^X | E \text{ has finite elements in } X\}$.

II. CONNECTEDNESS OF HYPERSPACE

Proposition 2.1 Let $X$ be topology space, then $(X)$ is dense in $2^X$.

Proof. For given any $U \in T, U \neq \emptyset$, we have $U$ contains the finite subset $n(X)$, and $(X) = \bigcup_{n=1}^{\infty} n(X)$, thus $\langle U \rangle \cap (X) \neq \emptyset$. Similarly, suppose $U_1, U_2, \ldots, U_n$ are nonempty open sets, $x_k \in U_k, (1 \leq k \leq n)$, then $\{x_1, x_2, \ldots, x_n\} \in \langle X, U_1 \rangle \cap \langle X, U_2 \rangle \cap \cdots \cap \langle X, U_n \rangle$.

Lemma 1 Let $X$ be topology space, we define a mapping $i: X \rightarrow 2^X$, $i(x) = \{x\}$, and then $i$ is continuous mapping.

Proof. Suppose $U \in T, U \neq \emptyset$, then

$i^{-1}(\langle U \rangle) = \{x \in X | i(x) \in U\} = \{x \in X | \{x\} \in U\} = U$.

If $U_1, U_2, \ldots, U_n \in T$, $i^{-1}(\langle U \rangle)$

$= \{x \in X | i(x) \in \bigcap_{i=1}^n < X, U_i > \neq \emptyset, 1 \leq i \leq n \}$

$= \{x \in X | x \in U_i, 1 \leq i \leq n \} = \bigcap_{i=1}^n U_i$.

Proposition 2.2 Let $X$ be topology space, a natural mapping $P_n: X^n \rightarrow \mathcal{F}_n(X)$, we define $P_n((x_1, \ldots, x_n))$
Let \( X = \{x_1, \cdots, x_n\} \), then \( P_r \) is continuous mapping.

Proof. For given any \( U \in T, U \neq \emptyset \), we have
\[
P^{-1}(<U>) = \{ (x_1, x_2, \cdots, x_n) \mid (x_1, x_2, \cdots, x_n) \in U^n \}
\]
where \( X_i = X_i, 1 \leq i \leq n, \) then \( P_r \) is a continuous mapping.

**Lemma 2** [2] Let \( X \) be topology space, suppose \( A \subset X \) is a closed (or an open) set, then \( \{ E \in 2^X \mid E \subset A \} \) is a closed (or an open) set in \( 2^X \).

**Corollary 1** Let \( X \) be topology space, suppose \( A \subset X \) is a closed set, and then \( \{ E \in 2^X \mid E \cap A \neq \emptyset \} \) is closed in \( 2^X \).

Proof. Since \( A \) is closed in \( X \), \( X \setminus A = B \) is open in \( X \).

By Lemma 2, \( \{ E \in 2^X \mid E \subset B \} = \{ E \in 2^X \mid E \subset X \setminus A \} \)
\[
= \{ E \in 2^X \mid E \cap A = \emptyset \}
\]
It follows that
\[
\{ E \in 2^X \mid E \cap A = \emptyset \} = 2^X \setminus \{ E \in 2^X \mid E \cap A = \emptyset \}
\]
is closed in \( 2^X \).

**Proposition 2.3** \( X \) is a connected topology space if and only if \( (X) \) is connected.

Proof. Let \( P_r : X^n \rightarrow n(X) \) be natural mapping, that is, \( P_r((x_1, \cdots, x_n)) = \{x_1, \cdots, x_n\} \). According to Lemma 1, \( P_r \) is a continuous mapping. As \( X \) is connected, \( X^n \) is connected. So \( n(X) \) is connected.

Since \( (X) = \bigcup_{n=1}^{\infty} n(X) \), \( (X) \) is connected.

**Proposition 2.4** [3] \( X \) is a connected topology space if and only if \( 2^X \) is connected.

Proof. Suppose \( X \) is connected, by [1]. \( X^n \) is connected, \( n = 1, 2, \cdots \). According to Proposition 2.2,
\[
P^{-1}(<U>) = \{ (x_1, x_2, \cdots, x_n) \mid (x_1, x_2, \cdots, x_n) \in U^n \}
\]
then \( P_r(n(X)) = n(X) \), then \( n(X) \) is connected, \( n = 1, 2, \cdots \).

\( (X) = \bigcup_{n=1}^{\infty} n(X) \) and \( \bigcap_{n=1}^{\infty} n(X) = 1(X) \neq \emptyset \), then \((X)\) is connected. Therefore the closure of \((X)\) is connected in \( 2^X \).

Suppose \( 2^X \) is connected, and \( X = \bigcup_{E \in 2^X} E \) is not connected, there exists nonempty sets \( A, B \) which is open and closed sets, such that \( X = A \cup B \) and \( A \cap B = \emptyset \), hence \( 2^X = \{ E \in 2^X \mid E \cap A \neq \emptyset \} \cup \{ E \in 2^X \mid E \cap A = \emptyset \} \).

Since \( A \) is closed, \( \{ E \in 2^X \mid E \cap A \neq \emptyset \} \) is closed in \( 2^X \). Since \( B \) is closed, \( \{ E \in 2^X \mid E \cap A = \emptyset \} \)
\[
= \{ E \in 2^X \}
\]
\[E \subset B\]
is closed in \( 2^X \).

Obviously,
\[
\{ E \in 2^X \mid E \cap A \neq \emptyset \} \cap \{ E \in 2^X \mid E \cap A = \emptyset \} = \emptyset \), \( \{ E \in 2^X \mid E \cap A \neq \emptyset \} \neq \emptyset \), \( \{ E \in 2^X \mid E \cap A = \emptyset \} = \emptyset \),

thus \( 2^X \) is not connected. This is contradiction. Therefore \( X \) is connected.

**Lemma 3** Suppose \( U_i \subset X, i = 1, 2, \cdots, n \) is connected, \( <U_1, U_2, \cdots, U_n> \) is connected.

Proof. Since \( U_i, i = 1, 2, \cdots, n \) is connected, \( (U_i) \) is connected (Theorem 4.10 of [1]).

\[ (U_1) \times (U_2) \times \cdots \times (U_n) \] is connected, and
\[
(X) \cap <U_1, U_2, \cdots, U_n> \text{ is under continuous mapping image of } (U_1) \times (U_2) \times \cdots \times (U_n),
\]
\[(X) \cap <U_1, U_2, \cdots, U_n> \text{ is connected},\]
\[(X) \cap <U_1, U_2, \cdots, U_n> \subset <\overline{U_1}, \overline{U_2}, \cdots, \overline{U_n}>, \text{ then } <U_1, U_2, \cdots, U_n> \text{ is connected.}\]

**Proposition 2.5** Suppose \( X \) is locally connected topology space, then \( 2^X \) is locally connected.

Proof. Suppose \( X \) is a locally connected topology space and \( E \in 2^X \), there exists a neighborhood \( V \) of \( E \) in \( 2^X \), we can find the connected open sets \( U_1, U_2, \cdots, U_n \subset T \) such that \( E \in <U_1, U_2, \cdots, U_n> \subset V \), hence \( 2^X \) is locally connected.

Suppose \( X \) is locally connected topology space, \( x \in U \subset X \), there exists a connected neighborhood \( \beta \) of \( \{x\} \), such that \( \beta \subset <U> \), so \( V = \bigcup_{A \in \beta} A \) is a
neighborhood of \( X \), \( V \subset U \) and \( \beta \) are connected. Therefore \( \{ x \} \in \beta \) and \( \{ x \} \) are connected.

**Lemma 4** [4] Suppose \( \beta \) is an open (closed) in \( 2^X \), then \( \bigcup_{E \in \beta} E \) is an open (closed) in \( X \).

**Lemma 5** Suppose \( U \) is a connected component, \( U \) is a connected closed set.

Proof. Since \( U \) is a connected component in \( X \), \( U \) is connected [1]. \( \overline{U} \) is a connected, \( U \subset \overline{U} \), \( U \) is a component in \( X \), then \( \overline{U} \) is a maximum connected set, \( \overline{U} = \overline{U} \), thus \( U \) is a connected closed set.

**Lemma 6** Suppose \( X \) is a locally connected topology space, then \( \overline{U} \) is a connected component in \( X \) and \( U \) is an open set.

Proof. Suppose \( P \in U \), \( X \) is a locally connected, \( P \) belong to an open connected set \( G_P \) at least, \( U \) is a component which contain \( P \), then \( P \in \bigcap G_P \subset U \) and \( U = \bigcup \{ G_P | P \in U \} \), thus \( U \) is open set as it is the union of open sets.

**Proposition 2.6** Suppose \( X \) is a locally connected topology space, \( U \) is a connected component in \( X \) and only if \( \{ E \in 2^X \mid E \subset U \} \) is a connected component in \( 2^X \).

Proof. Since \( X \) is locally connected, \( U \) is a connected component in \( X \). According to Lemma 2, it follows that \( \{ E \in 2^X \mid E \subset U \} \) is an open and closed set in \( 2^X \). Hence \( \{ E \in 2^X \mid E \subset U \} \) is a connected component in \( 2^X \).

Suppose \( \{ E \in 2^X \mid E \subset U \} \) is a connected component in \( 2^X \). Since \( X \) is locally connected, by Corollary 2, we have \( 2^X \) is locally connected, so \( \{ E \in 2^X \mid E \subset U \} \) is an open and closed set in \( 2^X \). We have \( U \in 2^X \):

In fact, \( U_1 = \bigcup \{ E \in 2^X \mid E \subset U \} \). If \( U \neq U_1 \), there exists \( x \in U \setminus U_1 \) such that \( \{ x \} \subset U \) and \( \{ x \} \in 2^X \), hence \( x \in U_1 \), this is contraction. So \( U = \bigcup \{ E \in 2^X \mid E \subset U \} \).

As \( \{ E \in 2^X \mid E \subset U \} \) is closed in \( 2^X \), by Lemma 3, \( U \) is closed in \( X \) and \( U \in 2^X \).

Since \( \{ E \in 2^X \mid E \subset U \} \) is an open and closed set in \( 2^X \), by Lemma 3, it follows that \( U \) is an open and closed set in \( X \), hence \( U \) is a connected component in \( X \).

**Lemma 7** [4] Let \( X, Y \) be topology space and \( X \) is path connected, \( f : X \to Y \) is continuous mapping, then \( f(X) \) is path connected.

**Proposition 2.7** Let \( X \) be topology space, then \( (X) \) is path connected.

Proof. Since \( X \) is path connected, \( X^n \) is path connected.

We define a natural mapping \( P_r : X^n \to n(X) \). By Proposition 2.2, \( P_r \) is a continuous mapping.

By Lemma 7, \( n(X) = P(X^n) \) is path connected, we have \( 1 \subset 2 \subset \cdots \subset n \subset \cdots \).

As \( (X) = \bigcup_{n=1}^{\infty} n(X) \), \( \forall E_1, E_2 \in (X) \), there exists \( n, m \in \mathbb{N}, E_1 \in n, E_2 \in m \).

Assume \( n \leq m, E_1 \in n \subset m \). Since \( m \) is path connected \([6]-[9]\), there exists a path from \( E_1 \) to \( E_2 \), then \( X \) is path connected.

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