Abstract—A multiscale finite element simulation is presented to solve a parabolic problem efficiently. This parabolic model has highly oscillatory coefficient, which make the conventional methods expensive costs or bad behaviors. The new multiscale finite element scheme, whose multiscale basis functions may reflect the local strong oscillation, can acquire good simulation on the macroscopical scale. The Euler backward difference time discretization is applied. Our method just computes on the coarse grids and saves plenty of computer resources, and it obtains convergent and controllable errors with the time iterations.

Keywords—multiscale finite element method; parabolic model; singular perturbation; high oscillation; Euler backward difference

I. INTRODUCTION

Parabolic problems with high oscillation are widely appeared in fluid mechanics, molecular dynamics, heat transfer and image processing, etc[1]. They have small perturbed parameters, which cause the boundary layers and strong oscillations in the problem.

Their highly-resolved solutions may be computationally expensive or even infeasible due to the great number of fine cells, so conventional methods cannot solve them effectively. As such, some reduction techniques are particularly interested in order to decrease the computational cost, and at the meantime, to acquire the good accuracy.

Hou and Wu[2] proposed the multiscale finite element method(MsFEM) initially, by solving the local problem to get the multiscale basis functions. After that, the multiscale computations have made many progresses[3-9]. Efendiev, Galvis and Hou[6] presented the generalized multiscale finite element method(GMsFEM) to construct the offline and online space, and extended MsFEM to more general problems. This paper author Sun[8] applied the multiscale basis functions for the singular perturbation on graded meshes, and it was shown to reduce the boundary layer errors remarkably. Jiang, Presho and Huang[9] studied an adapted Petrov-Galerkin MsFEM, which provided the more flexibilities in constructing the function spaces and eliminated the cell resonance effect to improve its convergence and accuracy.

The multiscale parabolic problem is hot and hard spots in recent scientific computation. We consider an initial and boundary value problem(IBVP):

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (k_x(x) \nabla u) &= f, & \Omega \in (0,1) \times (0,1) \cup t \in (0, T], \\
u(x, t) &= g, & \partial \Omega \cup t \in (0, T]; \\
u(x, 0) &= u_0, & \Omega \cup t = 0,
\end{align*}
\]

where \( u \) is the solution, \( t \) is time, \( k_x(x) \) is the highly oscillatory coefficient and is dependent on the scale parameter \( \varepsilon \), \( x \) is the two-dimensional space variable, \( f \) is right-side, \( g \) is boundary, \( u_0 \) is initial value, and \( \Omega, \partial \Omega \) are the domain and its boundary. This problem is well-posed under some conditions, but its analytical expression is unavailable. So the numerical and accurate solution is our goal in this paper.

II. STANDARD FEM AND MULTISCALE FEM FOR PARABOLIC PROBLEM

A. Variational Form and Finite Element Scheme

The variational form of (1) is to seek \( \mathbf{u} \in H^1(\Omega) \) such that

\[
\left( \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + a(u, v) = (f, v), \quad \forall \mathbf{v} \in H^1(\Omega), t \in (0, T],
\]

where \( a(u, v) = \int_{\Omega} k_x(x) \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \, dx \), \( (f, v) = \int_{\Omega} f \, dx \), and \( H^1 \) is the first order differentiable and square-integrable space. The corresponding matrix form is

\[
M \frac{du}{dt} + Au = F,
\]

where its element \( M_{ij} = \int_{\Omega} \phi_i \phi_j \, dx \), \( A_{ij} = \int_{\Omega} k_x(x) \nabla \phi_i \cdot \nabla \phi_j \, dx \), \( F_i = \int_{\Omega} f \phi_i \, dx \), and \( \phi_i \) is the standard finite element basis function.
We divide the domain \( \Omega \) into several sub-domain \( \Omega_k \), with \( \bigcup \Omega_k = \Omega \), and let \( K^k \) to be the rectangular uniform grids partition, thus in each element \( [x_i, x_{i+1}] \times [y_j, y_{j+1}] \) is the corresponding nodal coordinates and four standard basis functions are \( \phi_1 = (1-\xi)(1-\eta), \phi_2 = \xi(1-\eta), \phi_3 = \xi\eta, \phi_4 = (1-\xi)\eta \), where the variable \( \xi = (x-x_i)/h_x \), \( \eta = (y-y_j)/h_y \), mesh size \( h_x = x_{i+1} - x_i \), \( h_y = y_{j+1} - y_j \), and \( dx = h_x d\xi \), \( dy = h_y d\eta \). In this way, the local mass matrix element \[
M_{ij} = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \phi_i \phi_j \, dx \, dy = \int_0^1 \int_0^1 (1-\xi)^2(1-\eta)^2 h_x h_y \, d\xi \, d\eta = \frac{1}{9} h_x h_y ,
\]
and \( M_{12}, \ldots, M_{44} \) are obtained similarly. The local stiffness matrix element \[
A_{ij} = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial \eta} \frac{\partial \phi_j}{\partial \eta} \right) \, dx \, dy = \int_0^1 \int_0^1 \left( k_h \phi_i + \cdots + k_{4h} \phi_4 \right) \left( (1-\xi)^2 \frac{h_x}{h_y} + (1-\xi)^3 \frac{h_x}{h_y} \right) d\xi d\eta,
\]
and \( A_{12}, \ldots, A_{44} \) are similar. The local force vector element \[
F_i = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f \phi_i \, dx \, dy = \int_0^1 \int_0^1 \left( f_1 \phi_1 + \cdots + f_4 \phi_4 \right) \phi_i h_x h_y d\xi d\eta = \left( \frac{1}{9} f_1 + \frac{1}{18} f_2 + \frac{1}{36} f_3 + \frac{1}{18} f_4 \right) h_x h_y ,
\]
and \( F_2, F_3, F_4 \) are similar. In turn, we solve the global matrix system to acquire the Galerkin finite element solution \( u_h \).

### B. Multiscale Finite Element Scheme

Now we present the new multiscale FEM to settle the parabolic model. The corresponding variational form is to seek \( u_h \in U^h \) such that

\[
\left( \frac{\partial u_h}{\partial t}, v \right) + a(u_h, v) = (f, v), \quad \forall \, v \in U^h, \, t \in (0, T),
\]  

where \( u_h \) is the MsFEM solution, and the multiscale space \( U^h = \text{span} \{ \phi_i, \forall K \in K^h \} \) is built by the multiscale basis functions \( \phi_i \).

Being different from the standard bases \( \varphi_i \), the multiscale bases \( \phi_i \) are not expressed analytically. They are obtained from the sub-problem on coarse element \( K \)

\[
\begin{cases}
-\nabla \cdot (k(x) \nabla \phi) = 0, & \text{in } K; \\
\phi(x) = l_i, & \text{on } \partial K.
\end{cases}
\]  

Set the local boundary condition, i.e., \( \phi(x_j, y_j) = 1 \) when \( i = j \), \( \phi(x_j, y_j) = 0 \) when \( i \neq j \), and \( \phi(x_j, y_j) \) is varying linearly on \( \partial K \). Since the sub-problem (4) has the same differential operator as the original model (1), thus the multiscale basis functions \( \phi_i \) can show the local information such as the oscillations, which is not available from the former standard bases \( \varphi_i \).

### C. Backward Difference Time Discretization

Now we make the time discretization. We know that the Euler forward difference does not work, so here we outline the Euler backward difference for the model. Note that this backward difference is an inexplicit scheme.

\[
M \frac{u^{(k+1)} - u^{(k)}}{\Delta t} + Au^{(k+1)} = F
\]

\[
\Rightarrow (u^{(k+1)} - u^{(k)}) + \Delta t M^{-1} A u^{(k+1)} = \Delta t M^{-1} F
\]

\[
\Rightarrow u^{(k+1)} = (I + \Delta t M^{-1} A)^{-1} u^{(k)} + (I + \Delta t M^{-1} A)^{-1} \Delta t M^{-1} F
\]

\[
= (M + \Delta t A)^{-1} M \cdot u^{(k)} + \Delta t (M + \Delta t A)^{-1} F,
\]

Set \( S = (M + \Delta t A)^{-1} M, \, G = \Delta t (M + \Delta t A)^{-1} F \), then

\[
u^{(k+1)} = S \cdot u^{(k)} + G.
\]

Thus in this iterative way we can acquire the multiscale parabolic solution \( u_h^{(k+1)} \) along with the time eclipsing.

### III. Numerical Experiment

In this section, we use the standard FEM and the multiscale FEM to settle the parabolic model (1), respectively. The codes are compiled by Matlab2012. We set the domain \( \Omega \in (0,1) \times (0,1) \), and the coarse element partition number \( N_x = N_y = 8 \), in each coarse element the fine partition number \( n_x = n_y = 8 \). Note that the standard FEM computes on the fine grids \( (N_x \times n_x + 1) \times (N_y \times n_y + 1) = N_{\text{global nodes}} \), while the multiscale FEM just computes on the coarse grids \( (N_x + 1) \times (N_y + 1) = N_{\text{coarse nodes}} \). The latter method is shown to provide good accuracy, at the same time it saves the required computer memory and time.
We take the very small parameter $\varepsilon$ in the benchmark model from Stanford university permeableness coefficient $k_\varepsilon(x)$, see Figure I. And the right side $f = 1$, boundary $g = 0$, initial value $u_0 = \sin \pi x \cdot \sin \pi y$.

![Figure I. The highly oscillatory coefficient and its contour](image1)

The absolute error and relative error of $L^2$ norm are as followed

$$
\|u - u_h\|_{L^2} = \left( \int_\Omega (u - u_h)^2 \, dx \, dy \right)^{1/2},
$$

$$
\|u - u_h\|_{L^2} = \left( \int_\Omega (u - u_h)^2 \, dx \, dy \right)^{1/2}
\|u\|_{L^2} = \left( \int_\Omega u^2 \, dx \, dy \right)^{1/2}.
$$

Since we don’t have exact solution for the model (1), we take the FEM solution on very fine grid as the “exact” solution, and $u_h$ is the multiscale solution.

Set the time step $\Delta t = 0.001$, and iteration number is 30. After iterations, the standard FEM solution $u^{(30)}_g$ and the multiscale FEM solution $u^{(30)}_h$ are very similar, see Figure II. While we should keep in mind that the latter multiscale method is obtained from the coarse grid computations, and it just costs small amount of computer resource and time to acquire the good simulation.

![Figure II. Standard FEM solution $u^{(30)}_g$ and MSEFM solution $u^{(30)}_h$, respectively](image2)

![Figure III. Absolute error and relative error of L2 norm by MSFEM](image3)
From Figure III we provide the absolute error and relative error of $L^2$ norm by our MsFEM with the time eclipsing, and this means the numerical solution behaves well to present the stable and convergent approximations.

IV. CONCLUSIONS

In this paper, we present a multiscale simulation to solve a parabolic model with highly oscillatory coefficient. The multiscale finite element scheme and Euler backward time discretization are applied. With the help of the multiscale basis functions, we save computer costs to get good and convergent simulations for the parabolic problem.

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