The Characterization for Symmetric Tight Frames and Periodic Gabor Frames and Applications in Computer Engineering

Deyou Yuan

College of Math. & Physics, Nanyang Institute of Technology, Nanyang 473004, China

zas123qwe@126.com

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Abstract. Materials science is an interdisciplinary field applying the properties of matter to various areas of science and engineering. In this paper, we discuss a new set of symmetric tight frame wave -lets with the associated filterbanks outputs downsampled by several generators. The frames consist of several generators obtained from the lowpass filter using spectral factorization, with lowpass fil -ter via a simple approach using Legendre polynomials. The filters are feasible to be designed and offer smooth scaling functions and frame wavelets. We shall give an example to demonstrste that so-me examples of symmetric tight wavelet frames with three compactly supported real-valued sym-metric generators will be presented to illustrate the results.

Introduction

Manufacturing engineering is an interdisciplinary field applying the properties of matter to var -ious areas of science and engineering. Mechanical engineers apply the principles of mechanics and energy to the design of machines and devices: Energy and Motion. The frame theory has been one of powerful tools for researching into wavelets. Duffin and Schaeffer introduced the notion of frames for a separable Hilbert space in 1952. Later, Daubechies etal revived the study of frames in[1,2], and since then, fra -mes have become the focus of active research, both in theory and in applications, such as signal processing, image processing and sampling theory. The rise of frame theory in applied mathematics is due to the flexibility and redundancy of frames, where robustness, error tolerance and noise suppression play a vital role [3,4]. The concept of frame multiresolution analysis (FMRA) as described in [2] generalizes the notion of MRA by allowing non-exact affine frames. However, subspaces at different resolutions in a FMRA are still generated by a frame formed by translates and dilates of a single function. This paper is motivated from the observation that standard methods in sampling theory provide examples of multiresolution structure which are not FMRAs. Inspired by [2] and [5], we introduce the notion of a generalized multiresolution structure(BGMS) of $L^2(R^2)$, which has a pyramid decomposition scheme. It also lead to new constructions of affine frames of $L^2(R^2)$.

Fundamental Properties of Gabor Frames

Let W be a separable Hilbert space and Λ is an index set. We recall that a sequence $\{\lambda_v : v \in Z\} \subseteq W$ is a frame for H if there exist positive real numbers B, C such that

$$\forall\ \hbar \in W, \qquad B\|\hbar\|^2 \le \sum_{v\in\Lambda} \left|\langle \hbar, \lambda_v \rangle\right|^2 \le C\|\hbar\|^2 \tag{1}$$

A sequence $\{\lambda_v : v \in Z\} \subseteq W$ is a Bessel sequence if (only) the upper inequality of (1) holds. If only for all $\hbar \in \Omega \subset W$, the upper inequality of (1) holds, the sequence $\{\lambda_v\} \subseteq W$ is a Bessel sequence with respect to (w.r.t.) Ω. If $\{f_v\}$ is a frame, there exists a dual frame $\{f_v^*\}$ such that

$$\forall \Upsilon \in W\ ,\quad \Upsilon = \sum_{v\in\Lambda} \left\langle \Upsilon, f_v \right\rangle f_v^* = \sum_{v\in\Lambda} \left\langle \Upsilon, f_v^* \right\rangle f_v \tag{2}$$

To state our results, the Fourier transform of an integrable function $f(x) \in L^1\left(R^2\right)$ is defined by

$$Ff(\omega) = \hat{f}(\omega) = \int_{R^2} f(x) e^{-2\pi i x\omega} dx, \quad \omega \in R^2 \tag{3}$$

which, as usual, can be naturally extended to functions in $L^2(R^2)$. For a sequence $c = \{c(v)\} \in \ell^2(Z)$, we define its discrete-time Fourier transform as the function in $L^2(0,1)^2$ by

$$Fc(\omega) = C(\omega) = \sum_{v\in Z^2} c(v)\, e^{-2\pi i x\omega} dx \tag{4}$$

For any $f(t) \in L^2(R)$, the novel fractional Fourier transformis defined to be

$$F_\alpha(u) = \mathscr{F}^\alpha\{f(t)\}(u) = \int_{\mathbb{R}} f(t)\mathcal{K}_\alpha(u,t)dt$$

$$\psi_{\alpha,m,n}(t)=a_0^{-m/2}\psi\left(a_0^{-m}t - nb_0\right)e^{-i[(t^2-(na_0^m b_0)^2)/2]\cot\alpha}$$

A system $\{g_{m,n}\} = \{T_{na}E_{mb}g\}$ is a *Gabor frame* or *Weyl-Heisenberg frame* for $L^2(R^2)$, if there exist two constants $A, B > 0$ such that

$$A\|\hbar\|^2 \le \sum_{m,n\in Z^2} \left|\left\langle \hbar, g_{m,n}\right\rangle\right|^2 \le B\|\hbar\|^2$$

holds for all $\forall\ \hbar \in L^2(R^2)$. For a Gabor frame $\{g_{m,n}\}$ the *analysis mapping* (also called Gabor transform) U_g, given by $U_g : f \to \{< f, g_{m,n} >\}_{m,n}$, $\forall\ f \in L^2(R^2)$ and its adjoint, the *synthesis mapping* (also called Gabor expansion) U_g^*, given by

$$U_g^* : \{c_{m,n}\} \to \sum_{m,n} c_{m,n} g_{m,n}, \quad \forall\ \{c_{m,n}\} \in \ell^2(Z^2)$$

are bounded linear operators. The *Gabor frame operator* S_g is defined by $S_g = U_g^* U_g$. Explicitly,

$$S_g f \to \sum_{m,n} < f, g_{m,n} > g_{m,n}, \quad \forall\ f \in L^2(R^2)$$

If $\{g_{m,n}\}$ forms a Gabor frame for $L^2(R^2)$, then $\forall\ f \in L^2(R^2)$ can be written as

$$f = \sum_{m,n} < f, g_{m,n} > h_{m,n} = \sum_{m,n} < f, h_{m,n} > g_{m,n}, \tag{5}$$

where $h_{m,n}$ are the elements of the dual frame, given by $h_{m,n} = S^{-1} g_{m,n}$. Equation (5) provides a constructive answer how to recover f from its Gabor transform $\{< f, g_{m,n} >\}_{m,n}$ for given analysis window g and how to compute the coefficients in the series expansion $f = \sum_{m,n\in Z} C_{m,n} g_{m,n}$ for given atom g. The key is the corresponding dual frame $\{< f, g_{m,n} >\}_{m,n}$. A detailed analysis of Gabor frames brings forward some features that are basic for a further understanding of Gabor analysis. Most of these features are not shared by other frames such as wavelet frames.

Bivariate Generalized Multiresolution Structure

To characterize such a BGMS, we first introduce the concept of pseudoframes of translates.

Definition 1. Let $\{T_{va}\Upsilon,\ v \in Z^2\}$ and $\{T_{va}\widetilde{\Upsilon},\ v \in Z^2\}$ be two sequences in $L^2(R^2)$. Let U be a closed subspace of $L^2(R^2)$. We say $\{T_{va}\Upsilon,\ v \in Z^2\}$ forms an affine pseudoframe for U with respect to $\{T_{va}\widetilde{\Upsilon},\ v \in Z^2\}$ if

$$\forall\ f(x)\in U\,,\ \ f\left(x\right)=\sum\nolimits_{v\in Z}\left\langle f, T_{va}\widetilde{\Upsilon}\right\rangle T_{va}\Upsilon\left(x\right) \tag{6}$$

Define an operator $K:U\to\ell^2(Z^2)$ by

$$\forall\ f\left(x\right)\in U\,,\quad Kf=\{<f, T_{va}\Upsilon>\}\,, \tag{7}$$

and define another operator $F:\ell^2(Z^2)\to W$ such that

$$\forall c=\{c(v)\}_{v\in Z^2}\in\ell^2(Z^2)\,.\quad Fc=\sum\nolimits_{v\in Z^2}c(v)T_{va}\widetilde{\Upsilon}\,. \tag{8}$$

Theorem 1. Let $\left\{T_{va}\Upsilon\right\}_{v\in Z^2}\subset L^2(R^2)$ be a Bessel sequence w.r.t. the subspace $U\subset L^2(R^2)$, and $\{T_{va}\widetilde{\Upsilon}\}_{v\in Z^2}$ is a Bessel sequence in $L^2(R^2)$. Assume K be defined by (7), and F be defined by(8). Assume that P is a projection from $L^2(R^2)$ onto U. Then $\left\{T_{va}\Upsilon\right\}_{v\in Z^2}$ is pseudoframes of translates for U with respect to $\{T_{va}\widetilde{\Upsilon}\}_{v\in Z^2}$ if and only if

$$KFP=P\,. \tag{9}$$

Proof. The convergence of all summations of (7) and (8) follows from the assumptions that $\left\{T_{va}\Upsilon\right\}_{v\in Z^2}$ is a Bessel sequence with respect to the subspace Ω, and $\{T_{va}\widetilde{\Upsilon}\}_{v\in Z^2}$ is a Bessel sequence in $L^2\left(R\right)$, with which the proof of the theorem is direct forward.

$$\sum_{k\in Z^n}\left|\left\langle h,\varphi_{v,k}\right\rangle\right|^2+\sum_{\sigma=1}^{\tau}\sum_{k\in Z^n}\left|\left\langle h,\psi_{v,k}^{\sigma}\right\rangle\right|^2$$

$$=\frac{1}{(2\pi)^n}\int_{[-3^{v-1},3^{v-1}]^n}\left|\sum_{k\in Z^n}\hat{h}(\omega+2\pi3^v k)\overline{\hat{\varphi}(3^{-v}(\omega+2\pi3^v k))}\right|^2 d\omega$$

$$+\frac{1}{(2\pi)^n}\sum_{\sigma=1}^{\tau}\int_{[-3^{v-1},3^{v-1}]^n}\left|\sum_{k\in Z^n}\hat{h}(\omega+2\pi3^v k)\overline{\hat{\psi}^{\sigma}(3^{-v}(\omega+2\pi3^v k))}\right|^2 d\omega$$

$$=\frac{1}{(2\pi)^n}\sum_{\sigma=0}^{\tau}\int_{[-3^{v-1},3^{v-1}]^n}\left|\sum_{\beta\in Z^n}\hat{h}(\omega+2\pi3^v k)\overline{s_\sigma(3^{-v-1}\omega+2\pi k/3)\hat{\varphi}(3^{-v-1}\omega+2\pi k/3)}\right|^2 d\omega$$

$$=\frac{1}{(2\pi)^n}\sum_{\sigma=0}^{\tau}\int_{[-3^{v-1},3^{v-1}]^n}\{\left|R_0(\omega)\overline{s_\sigma(3^{-v-1}\omega)}+R_1(\omega)\overline{s_\sigma(3^{-v-1}\omega+\eta_1)}\right.$$

$$\left.+R_2(\omega)\overline{s_\sigma(3^{-v-1}\omega+\eta_2)}+\cdots+R_{3^n-1}(\omega)\overline{s_\sigma(3^{-v-1}\omega+\eta_{3^n-1})}\right|^2\}d\omega$$

$$=\frac{1}{(2\pi)^n}\int_{[-3^{v-1},3^{v-1}]^n}\{\left|R_0(\omega)\right|^2+\left|R_1(\omega)\right|^2+\left|R_2(\omega)\right|^2+\sum_{j=3}^{3^n-1}\left|R_j(\omega)\right|^2\}d\omega$$

$$=\frac{1}{(2\pi)^n}\int_{[-3^{v-1},3^{v-1}]^n}\left|\sum_{k\in Z^n}\hat{h}(\omega+2\pi3^{v+1}k)\overline{\hat{\varphi}(3^{-v-1}\omega+2\pi k/3)}\right|^2 d\omega$$

$$+\frac{1}{(2\pi)^n}\int_{[-3^{v-1},3^{v-1}]^n}\left|\sum_{k\in Z^n}\hat{h}(\omega+2\pi3^{v+1}k+3^{v+1}\eta_1)\overline{\hat{\varphi}(3^{-v-1}\omega+2\pi k/3+\eta_1)}\right|^2 d\omega$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{1}{(2\pi)^n}\int_{[-3^{v-1},3^{v-1}]^n}\left|\sum_{k\in Z^n}\hat{h}(\omega+2\pi3^{v+1}k+3^{v+1}\eta_{3^n-1})\overline{\hat{\varphi}(3^{-v-1}\omega+2\pi k/3+\eta_{3^n-1})}\right|^2 d\omega$$

$$=\frac{1}{(2\pi)^n}\int_{[-3^{v},3^{v}]^n}\left|\sum_{k\in Z^n}\hat{h}(\omega+2\pi3^{v+1}k)\overline{\hat{\varphi}(3^{-v-1}\omega+2\pi k/3)}\right|^2 d\omega$$

$$=\sum_{k\in Z^n}\left|\left\langle h,\varphi_{v+1,k}\right\rangle\right|^2<+\infty.$$

$$W_f^{\alpha}(a,b) == \int_{\mathbb{R}} \sqrt{2\pi} F_{\alpha}(u) \widehat{\psi}^{*}(au \csc \alpha) \mathcal{K}_{-\alpha}(u,b) db$$

Theorem 2. Let $f(x), \widetilde{f}(x), \Upsilon_{\iota}(x)$ and $\widetilde{\Upsilon}_{\iota}(x)$, $\iota \in I$ be functions in $L^2(R^2)$ .Assume that conditions in Theorem 2 are satisfied. Then, for any funct- ion $\Gamma(x) \in L^2\left(R^2\right)$, and any integer n,

$$\sum_{k \in Z^2} \left\langle \Gamma, \widetilde{f}_{n,k} \right\rangle f_{n,k}(x) = \sum_{\iota=1}^{80} \sum_{v=-\infty}^{n-1} \sum_{k \in Z^2} \left\langle \Gamma, \widetilde{\Upsilon}_{\iota:v,k} \right\rangle \Upsilon_{\iota:v,k}(x) .$$

Furthermore, for any bivariate function $\Gamma(x) \in L^2\left(R^2\right)$, $\Gamma(x) = \sum_{\iota=1}^{80} \sum_{v=-\infty}^{+\infty} \sum_{k \in Z^2} \left\langle \Gamma, \widetilde{\Upsilon}_{\iota:v,k} \right\rangle \Upsilon_{\iota:v,k}(x)$. Consequently, if $\{\Upsilon_{\iota:v,k}(x)\}$ and $\{\widetilde{\Upsilon}_{\iota:v,k}(x)\}$, $(\iota \in \Lambda, v \in Z,\ \ k \in Z^2)$ are also Bessel sequences, they are a pair of affine frames for $L^2(R^2)$.

Conclusion

We characterize the pseudoframes of translates for the subspaces of $L^2\left(R^2\right)$. The pyramid decomposition scheme is derived based on such a BGMS. As a major new constribution the construction of affine frames for $L^2(R^2)$ based on a BGMS is presented.

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