The Study of Periodic Tight Framelets and Wavelet Frame Packets and Applications

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Abstract. Information science focuses on understanding problems from the perspective of the stake holders involved and then applying information and other technologies as needed. A necessary and sufficient condition is identified in term of refinement masks for applying the unitary extension principle for periodic functions to construct tight wavelet frames. Then a theory on the approximation order of truncated tight frame series is established, which facilitates construction of tight periodic wavelet frames with desirable approximation order. The pyramid decomposition scheme is derived based on the generalized multiresolution structure.

Introduction and Concepts

The setup of tight wavelet frames provides great flexibility in approximating and representing periodic functions. Fundamentals issues involved include the construction of tight periodic wavelet frames, approximation powers of such wavelet frames, and whether wavelet frames lead to sparse representat of locally smooth periodic functions. The frame theory plays an important role in the modern time-frequency analysis. It has been developed very fast over the last twenty years, espe cially in the context of wavelets and Gabor systems. This scientific field investigates the relationship betwenrse represeen the structure of materials at atomic or molecular scales and their macroscopic properties. Wavelet theory has been applied to signal processing, image compression, and so on. Frames for a separable Hilbert space were formally defined by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series. Basically, Duffin and Schaeffer abstracted the fundamental notion of Gabor for studying signal processing [2]. These ideas did not seem to generate much general interest outside of nonharmonic Fourier series however (see Young's [3]) until the landmark paper of Daubechies, Grossmann, and Meyer [4] in 1986. After this groundbreaking work, the theory of frames began to be more widely studied both in theory and in applications [5,6], such as signal processing, image processing, data compression, sampling theory.

We begin with the unitary extension pinciple and formulate a general procedure for constructing wavelet frames. The emphasis is on having refinement masks as the starting point. The condition for this can be easily verified and also provide insight to the refinement masks that enable the construction process. Let $L^2[0,2a]$ be the space of all $2a$ -periodic square-integrable complex-valued func tions over the real line R with inner product $<\cdot,\cdot>$ given by

$$<h,v>=1/(2a)\int_0^{2a} h(x)\overline{v(x)}\,dx$$

where $h(x),v(x)\in L^2[0,2a]$, and norm $\|u\|_2=<u,u>^{1/2}$. For a function $h(x)\in L^2[0,2a]$, we denote its Fourier series as $\sum_{n\in Z} h(n)e^{in}$, where $h(n)=<h,e^{in}>$, $n\in Z$, are its Fourier coefficient. For any $0\le u\in Z$, we define the shift operator $S_u^\ell : L^2[0,2a]\mapsto L^2[0,2a]$ by $S_u^\ell h(x):=h(x-2a\ell/2^u)$. For $h(x)\in L^2[0,2a]$, since $h(x)$ is a periodic function, it suffices to consider the shifts $S_u^\ell h(x), \ell\in\Lambda_k$, where Λ_k is given by

$$\Lambda_k := \{-2^{k-1}+1, -2^{k-1}+2, \cdots, 2^{k-1}-1, 2^{k-1}\}.$$

Let $\Delta(2^k)$ be the 2^k – periodic complex sequences b_k, that is $b_k(\ell+2^k u)=b_k(\ell)$ for all $\ell, u \in Z$. We denote the discrete Fourier transform of $b_k \in \Delta(2^k)$ by $\hat{b}_k(n) := \sum_{\ell\in\Omega_k} b_k(\ell)e^{-2ai\ell n/2^k}$. The sequence $\hat{b}_k$ also lies in $\Delta(2^k)$. Now consider positive integers $\eta_k, k \ge 0,$ and functions $\varphi_0, \psi_k^r, k \ge 0, r = 1, 2, \cdots, \eta_k$ in $L^2[0, 2a]$. The set $\{\varphi_0\}\bigcup\{S_k^\ell \psi_k^r, k \ge 0, r = 1, 2, \cdots, \eta_k, \ell \in \Lambda_k\}$ forms a normalized tight wavelet frame, or simply by tight wavelet frame for the space $L^2[0, 2a]$ if

$$\|h\|^2 = |< h, \varphi_0 >|^2 + \sum_{k=0}^{+\infty}\sum_{r=1}^{\eta_k}\sum_{\ell\in\Omega_k} |< h, S_k^\ell \psi_k^r >|^2 .$$

Our construction of wavelets is based a sequence of refinable functions $\{\varphi_k\}_{k\ge 0}$ in $L^2[0, 2a]$, which satisfies for every $k \ge 0$, the periodic refinement equation

$$\varphi_k = \sqrt{2}\sum_{\ell\in\Omega_{k+1}} b_{k+1}(\ell) S_{k+1}^\ell \varphi_{k+1}$$

for some $b_{k+1} \in \Delta(2^{k+1})$. For each $k \ge 0$, the wavelets $\psi_k^r \in L^2[0, 2a],\ r = 1, 2, \cdots, \eta_k$ with η_k be some positive integer, are given by the periodic refinement equation

$$\psi_k^r = \sqrt{2}\sum_{\ell\in\Omega_{k+1}} d_{k+1}^r(\ell) S_{k+1}^\ell \varphi_{k+1}$$

where $d_{k+1}^r \in \Delta(2^{k+1}),\ , r = 1, 2, \cdots, \eta_k$.Suppose that V is a separable Hilbert space. We recall that a sequence $\{f_k,\ k \in Z\} \subset V$ is a frame for V, if there exist positive real numbers A_1, A_2 such that

$$\forall\ \lambda \in V, \quad A_1\|\lambda\|^2 \le \sum_v \left|\left\langle \lambda, f_k\right\rangle\right|^2 \le A_2\|\lambda\|^2 \quad . \tag{1}$$

A sequence $\{f_k,\ k \in Z\} \subset V$ is a Bessel sequence if only the upper inequaliry of (1) holds. If only for all $\{f_k,\ k \in Z\} \subset \Omega \subset V$,the upper inequality of (1) follows the sequence $\{f_k\} \subset V$ is a Bessel sequence with respect to (w.r.t.) Ω. If $\{f_k\} \subset V$ is a frame there exist a dual frame $\{f_k^*,\ k \in Z\} \subset V$ such that

$$\forall \xi \in \Omega, \quad \xi = \sum_v \left\langle \xi, f_v\right\rangle f_v^* = \sum_v \left\langle \xi, f_v^*\right\rangle f_v \tag{2}$$

The following example gives frame-like decompositons for a subspace that are not frames. Let $\{f_n^*\}$ be a Riesz basis for a unique (biorthogonal) dual sequence $\{f_n^0\} \subset \Omega$ such that $< f_n^0, f_k^* >= \delta_{n,k}$, $n, k \in Z$ and

$$\xi \in \Omega, \quad \xi = \sum_{v\in Z} \left\langle \xi, f_v^*\right\rangle f_v^0 = \sum_{v\in Z} \left\langle \xi, f_v^0\right\rangle f_v^* \tag{3}$$

In the context of biorthogonal bases within V ,this is usually the end of it. However, choose now a function $\Delta f_n \in \Omega^\perp \subset V$, and let $f_n = f_n^0 + \Delta f_n$ .It obviously follows that

$$\left\langle f_n, f_k^*\right\rangle = \left\langle f_n^0 + \Delta f_n, f_k^*\right\rangle = \left\langle f_n^0, f_k^*\right\rangle + 0 = \delta_{n,k} \tag{4}$$

and for all $\xi \in \Omega$, $\sum_{v\in Z}\left\langle \xi, f_v\right\rangle f_v^0 = \sum_{v\in Z}\left\langle \xi, f_v^0 + \Delta f_v\right\rangle f_v^* = \xi$. The conclusion is that there are infinitely many biorthogonal duals $\{f_k\}$ to a biorthogonal basis $\{f_k^*\}$ of subspace Ω, so long if $\{f_k\}$ is allowed to go beyond Ω.

Multiple Pseudoframe and Generalized Multiresolution Structure

Let $J=\{1,2,\cdots s\}$ be a finite index set where s be a positive constant integer. We consider the case of multiple generators, which yield multiple pseudoframes for subspaces of $L^2(R)$.

Definition 1. Let $\{T_k\Upsilon_j\}$ and $\{T_k\widetilde{\Upsilon}_j\}$ $(j\in J, k\in Z)$ be two sequences in V, We say $\{T_k\Upsilon_j\}$ forms a multiple pseudoframe for $\Omega\subset V$ with respect to (w.r.t.) $\{T_k\widetilde{\Upsilon}_j\}$, if

$$\forall\ f(x)\in\Omega,\quad f(x)=\sum_{j\in J}\sum_{k\in Z}\left\langle f,T_k\Upsilon_j\right\rangle T_k\widetilde{\Upsilon}_j\,. \tag{5}$$

where we define a translate operator, $T_k\Upsilon(x)=\Upsilon(x-k)$, for a function $\Upsilon(t)\in L^2(R)$.

Definition 2. A Generalized multiresolution structure (GMS) $\{U_k, f_j, \widetilde{f}_j\}$ is a sequence of closed linear subspaces $\{U_k\}$ of $L^2(R)$ and $2r$ elements $f_j,\widetilde{f}_j\in L^2(R)$ $(j\in J)$ such that (a) $U_k\subset U_{k+1}$; (b) $\bigcap_{k\in Z}U_k=\varnothing$, $\bigcup_{k\in Z}U_k=L^2(R)$; (c) $\hbar(x)\in U_k$ if and only if $D\hbar(x)\in U_{k+1}$, where the dilation operator $Df(x)=2^{1/2}f(2x)$, for $f(x)\in L^2(R)$; (d) $\hbar(x)\in U_0$ implies $\hbar(x-k)\in U_0$, for all $k\in Z$; (e) $\{T_k f_j\}$ forms a multiple pseudoframes for U_0 with respect to $\{T_k\widetilde{f}_j\}$ $(j\in J, k\in Z)$.

Construction of Affine Pseudoframes for Space $L^2(R)$

In order to split a function $f(x)$ of U_1 into two functions (mostly) in S_0 and W_0, respectively, we will construct an affine pseudoframe for U_0 making use of the existing affine pseudoframe structure for U_0. Conventional symbols, $\Upsilon_l(x)$ and $\widetilde{\Upsilon}_l(x)$,will be used as generating functions for W_0. But they need notbe contained in W_0.

Definition 3. Let $\{U_k, f(x), \widetilde{f}(x)\}$ be a given GMS, and let $\Upsilon_l(x)$ and $\widetilde{\Upsilon}_l(x)$ $(l\in J)$ be $2s$ band-passfunctions in $L^2(R)$. We say $\left\{T_v f(x), T_v\widetilde{\Upsilon}_l(x), l\in J\right\}$ forms an affine pseudoframe for U_1 with respect to $\left\{T_v\widetilde{f}(x), T_v\widetilde{\Upsilon}_l(x), l\in J\right\}$, if

$$\forall\Phi(x)\in U_1,\Phi(x)=\sum_{v\in Z}\left\langle\Phi,T_v\widetilde{f}\right\rangle T_v f(x)+\sum_{l\in J}\sum_{v\in Z}\left\langle\Phi,T_v\widetilde{\Upsilon}_l\right\rangle T_v\Upsilon_l(x)\,, \tag{6}$$

Accordingly, $\{T_{va}\widetilde{f},T_{va}\widetilde{\Upsilon}\iota,\ \iota\in J\}$ is called a dual affine pseudoframes with respect to $\{T_v f, T_v\Upsilon_\iota\}$ ($\iota\in J$) in the sense of (6).

Then there exists a bivariate function $f(x)\in L^2(R)$ (see ref.[3]) such that

$$f(x)=\sqrt{2}\sum_{v\in Z}d_0(v)f(2x-v)\,. \tag{7}$$

There exists a scaling relationship for function $\widetilde{f}(x)$ under the same conditions as that of d_0 for a sequence $\widetilde{d_0}$, that is

$$\widetilde{f}(x)=\sqrt{2}\sum_{s\in Z}\widetilde{d_0}(s)\widetilde{f}(2x-s)\,. \tag{8}$$

To characterize the condition for which $\left\{T_v f, T_v\Upsilon_l,\ l\in J\right\}$ forms an affine pseudoframe for

V_1 with respect to $\left\{T_v\widetilde{f}, T_v\widetilde{\Upsilon}_l,\, l\in J\right\}$, we begin with developing the wavelet refinement equations associated with band-pass functions $\Upsilon_l\left(l\in J\right)$ and $\widetilde{\Upsilon}_l\left(l\in J\right)$ based on a generalized multiresolution structure, namely,

$$\Upsilon_l(x)=\sqrt{2}\sum\nolimits_{v\in Z} m_l(v) f(2x-v), \quad l\in J \quad \text{in}\, L^2\left(R\right), \tag{9}$$

$$\widetilde{\Upsilon}_l(x)=\sqrt{2}\sum\nolimits_{v\in Z} \widetilde{m}_l(v) \widetilde{f}(2x-v), \quad l\in J \quad \text{in}\;\; L^2\left(R\right). \tag{10}$$

Let $\chi_\Lambda\left(\omega\right)$ be the characteristic function of the interval Λ defined in Proposition 1. We shall use the following 1-periodic function. $\Gamma_\Lambda\left(\omega\right)\equiv\sum\nolimits_k \chi_\Lambda\left(\omega+k\right)$ Theorem 2. Let Λ be the bandwidth of the subspace V_0 defined in Proposition 1. $\{T_v f, T_v\Upsilon_l,\, v\in Z,\, l\in J\}$ forms an affine pseudoframe for V_1 with respect to $\left\{T_v\widetilde{f}, T_v\widetilde{\Upsilon}_l, l\in J\right\}$ if and only if there exist D_0 and D_l ,$\left(l\in J\right)$ in $L^2\left(\left[0,1\right]\right)$) such that

$$\sum\nolimits_{l\in J} D_l\left(\omega\right)\overline{\widetilde{M}_l\left(\omega\right)}\Gamma_\Lambda\left(\omega\right)=2\Gamma_\Lambda\left(\omega\right) \tag{11}$$

$$\sum\nolimits_{l\in J} D_l\left(\omega+1/2\right)\overline{\widetilde{M}_l\left(\omega+1/2\right)}\Gamma_\Lambda\left(\omega\right)=0 \tag{12}$$

Theorem 1. Let $f(x), \widetilde{f}(x), \Upsilon_\iota(x)$ and $\widetilde{\Upsilon}_\iota(x), \iota\in J$ be functions in $L^2\left(R\right)$ defined by (7), (8), (9) and (10) respectively. Assume that conditions in Definition 1 are satisfied. Then, for any function $\Psi(x)\in L^2(R)$, and any integer n,

$$\sum_{k\in Z}\left\langle \Psi, \widetilde{f}_{n,k}\right\rangle f_{n,k}(x)=\sum_{\iota=1}^{s}\sum_{v=-\infty}^{n-1}\sum_{k\in Z}\left\langle \Psi, \widetilde{\Upsilon}_{\iota:v,k}\right\rangle \Upsilon_{\iota:v,k}(x). \tag{13}$$

Moreover, for any univariate function $\Psi(x)\in L^2(R)$, $\Psi(x)=\sum_{\iota=1}^{s}\sum_{v=-\infty}^{+\infty}\sum_{k\in Z}\left\langle \Psi, \widetilde{\Upsilon}_{\iota:v,k}\right\rangle \Upsilon_{\iota:v,k}(x)$. Consequently, if $\{\Upsilon_{\iota:v,k}(x)\}$ and $\{\widetilde{\Upsilon}_{\iota:v,k}(x)\}$, $(\iota\in J, v\in Z,\;\; k\in Z)$ are also Bessel sequences, they are a pair of affine frames for the space $L^2(R)$.

Conclusion

The multiple affine pseudoframes for the subspaces of $L^2\left(R\right)$ are studied. The pyramid decomposition scheme is derived based on such a GMS. As a major new con stribution the constructionof affine frames for $L^2\left(R\right)$ based on a GMS is presented.

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