Decomposing Complete 3-Uniform Hypergraph $K_n^{(3)}$ into 7-cycles

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Abstract—This paper respectively introduces the Jianfang Wang-Tony Lee and Katona-Kierstead definition of a Hamiltonian cycle in a uniform hypergraph. In general, the existence of decomposition into 7-cyces remains open. In this paper, researchers use the method of edge-partition and cycle sequence proposed by Jirimutu and Jianfang Wang. Researchers find a new result.

Keywords-Uniform Hypergraph; 7-cycle; Cycle Sequence; Cycle Decomposition; Hamiltonian Cycle

I. INTRODUCTION

Hypergraphs are subsets systems of finite sets and may be regarded as one of the most general structures in discrete mathematics. As a natural generalization of Hamiltonian decomposition of graphs, researchers have the problem of Hamiltonian cycle decomposition of hypergraphs. In 1970, the notion of Hamiltonicity was first generalized to uniform hypergraphs by Berge in [1]. His definition of a Hamiltonian cycles in a hypergraph $H=(V,E)$ is a sequence $(v_0,e_1,v_1,e_2,\cdots,v_{n-1},e_n,v_0)$, where $V=\{v_0,v_1,\cdots,v_{n-1}\}$ and $e_1,e_2,\cdots,e_n$ are distinct elements of E. $v_{i-1}, v_i \in e_i$ with addition of indices modulo r. A cycle of this type is called a Berge cycle. In 1970, the study of decomposition of complete 3-uniform hpergraphs into cycles of this type was begun by Bermond in [2]. In 1994, this problem was solved by Verrall in [3]. Many papers studied the two different definitions of a Hamiltonian cycle in [6] and [8], which are due to Katona-Kierstead Wang-Lee respectively. In fact, two different definitions of a Hamiltonian chain and Hamiltonian cycle are the same. A decomposition of the complete k-uniform hpergraphs into Hamiltonian cycles has been considered in [7] to [14]. In paper [7], Hamiltonian decompositions of $K_n^{(3)}$ for all admissible $n\le 32$ has been resolved. Recently, by programming, using the method of edge-partition and cycle sequence, researchers have obtained some results for all admissible $32<n\le 46$ and $n\ne 43$ in [15]. In [7], the problem of decomposing the complete 3-uniform hypergraph into ℓ-cycles $(\ell\ge 5)$ are open. Meszka and Rosa have introduced a necessary condition for the existence of such a decomposition is that $n\equiv 1,2,5,7,10\ or\ 11 \pmod{15}$. A decomposition of $K_n^{(3)}$ into 5-cycles exists for all admissible $n\le 17$, for all $n=4^m+1$, m is a positive integer. In paper [13], they find a decomposition of $K_n^{(3)}$ into 5cycles for $n\in\{5,7,10,11,16,17,20,22,26\}$ and has showed if $K_n^{(3)}$ can be decomposition into 5-cycles, then $K_{5n}^{(3)}$ can be decomposed into 5-cycles. In paper [18], researchers have found a decomposition of $K_n^{(3)}$ into 7-cycles for $n\in\{7,8,14,16,22,23\}$ and have shown if $K_n^{(3)}$ can be decomposition into 7-cycles, and then $K_{7n}^{(3)}$ can be decomposed into 7-cycles. In this paper, researchers find a decomposition of $K_{29}^{(3)}$ into 7-cycles.

II. PRELIMINARIES

A hypergraph $H=(V,E)$ consists of a finite set V of vertices with a family E of subsets of V, called hyperedges (or simply edges). If each (hyper)edge has size k, researchers say that H is a k-uniform hypergraph. In particular, the complete k-uniform hypergraph on n vertices has all k-subsets of $\{1,2,\cdots,n\}$ as edges, denoted this by $K_n^{(k)}$. The (hyper)edges of $K_n^{(3)}$ is denoted by $\varepsilon(K_n^{(3)})$.

Definition 2.1 Let $H=(V,E)$ be a k-uniform hypergraph. A l-cycle in H is a cyclic sequence

$(v_0, v_1, \cdots, v_{l-1})$ of V where $3 \le k < l-1$ such that each consecutive k-tuple of vertices is an edge of E.

Definition 2.2 A l-cycle decomposition of H is a partition of the set of (hyper)edges of H into mutually-disjoint l-cycles.

Researchers introduce the method of edge-partition and cycle-model in [10].

Definition 2.3[9] Let $T = \{a, b, c\}$ be a triple of distinct elements of $\mathbf{Z}_n$. Then its difference pattern $\pi(T)$, is the equivalence class of ordered triples containing cyclic rotations of $(b-a, c-b, a-c)$ and $(c-a, b-c, a-b)$ (where the differences are taken modulo n).

Clearly, the three differences sum to zero. Therefore if researchers know that the first two differences are x and y, then the third is $n-x-y$. Omit the third number, and researchers can get a difference pair. Using edge-partition of $K_n^{(3)}$ in paper [10], all difference pairs of $K_n^{(3)}$ may be obtained. Let $\mathbf{Z}_n$ be the set of integers, n be a fixed positive integer, and $\mathbf{Z}_n = (0, 1, \cdots, n-1)$.

Let $D_{all}(n) = \{(k_1, k_2) \mid 1 \le k_1, k_2 \le n-1, \text{and } k_1 + k_2 \ne n\}$, $D(n) = D_e \cup D_l \cup D_m$, where

$$D_e = \{(k_1, k_2) \in D_{all}(n) \mid k_1 = k_2 = k, 1 \le k < \frac{n}{2}\},$$

$$D_l = \{(k_1, k_2) \in D_{all}(n) \mid 1 \le k_1 < k_2 < \frac{n-k_1}{2}\},$$

$$D_m = \{(k_2, k_1) \in D_{all}(n) \mid (k_1, k_2) \in D_l\}.$$

Given a difference pair $(k_1, k_2) \in D_{all}(n)$ and an integer $m \in \mathbf{Z}_n$, define a subhypergraph of $K_n^{(3)}$ generated by (k_1, k_2) as follows:

$$E(m; k_1, k_2) = \{m, m+k_1, m+k_1+k_2\} (\text{mod } n),$$

denote by $H(k_1, k_2) = \{E(m; k_1, k_2) \mid m \in \mathbf{Z}_n\}$, where the addition is performed modulo n.

Lemma 2.4[10] Let (k_1, k_2) and (k_1', k_2') be arbitrary two distinct difference pairs in $D_{all}(n)$. Researchers have either $H(k_1, k_2) \cap H(k_1', k_2') = \varnothing$ or $H(k_1, k_2) = H(k_1', k_2')$, a necessary and sufficient condition for the second equation is

$$(k_1, k_2) = \begin{cases} (k_1', k_2') & or \\ (k_1'+k_2', -k_2') & or \\ (-k_1', k_1'+k_2') & or \\ (k_2', -k_1'-k_2') & or \\ (-k_1'-k_2', k_1') & or \\ (-k_2', -k_1') \end{cases} \quad (\text{mod } n)$$

Definition 2.5[10] Let (k_1, k_2) and (k_1', k_2') be arbitrary two distinct difference pairs in $D_{all}(n)$. Researchers say (k_1, k_2) and (k_1', k_2') are equivalent if $H(k_1, k_2) = H(k_1', k_2')$. This is denoted by $(k_1, k_2) \square (k_1', k_2')$. If not, (k_1, k_2) and (k_1', k_2') are inequitable.

Theorem 2.6[10] (Edge-partition of $K_n^{(3)}$)

For any $K_n^{(3)}$, $\varepsilon(K_n^{(3)}) = \bigcup_{(k_1, k_2) \in D(n)} H(k_1, k_2)$, where $(k_1, k_2) \in D(\text{n})$. If $k_1 \ne k_2$, for convenience, researchers use (k_1, k_2) denote (k_1, k_2) and (k_2, k_1).

Theorem 2.6 Let n be a positive integer. Construct $(k_1, k_2, \cdots, k_l)$ be a sequence on $D_{all}(\text{n})$, where for any $1 \le i, j \le l-1$, $(k_i, k_{i+1}) \in D_{all}(\text{n})$, (k_i, k_{i+1}) and (k_j, k_{j+1}) are inequitable when $i \ne j$. The sequence $(k_1, k_2, \cdots, k_l)$ induces the cycle sequence $(r_0, r_1, \cdots, r_{l-1})$. This sequence satisfies the following two conditions:

a) $r_0 = 0, \sum_{i=1}^{j} k_i \equiv r_j \ (\text{mod n}), r_l = 0$.

b) For any $i, j (i \ne j), r_i \ne r_j$.

Then $(r_0, r_1, \cdots, r_{l-1})$ is a l-cycle, denoted by $\mathrm{C}_l = (r_0, r_1, \cdots, r_{l-1})$ and called base cycle. According to the difference pattern $\pi(T)$, obviously, researchers obtained the set of l-cycles $\{\mathrm{C}_l + \mathrm{i} \mid \mathrm{i} \in \mathbf{Z}_n\}$. In particular, if $l = n$, $(r_0, r_1, \cdots, r_{n-1})$ is a base Hamiltonian cycle, denoted by $\mathrm{C}_n = (r_0, r_1, \cdots, r_{n-1})$.

Theorem 2.7 Let n be a positive integer, for any $1 \le i, j \le l-1$, $(k_i, k_{i+1}) \in D_{all}(\text{n})$, (k_i, k_{i+1}) and (k_j, k_{j+1}) are inequitable when $i \ne j$, obtained that

$(k_1, k_2, \cdots, k_l)$ be a sequence on $D_{all}(\mathrm{n})$ .Then $H(k_1, k_2, \cdots, k_l) = \bigcup_{i=1}^{l} H(k_i, k_{i+1})$,where $k_{l+1} = k_1$.

III. DECOMPOSING $K_{29}^{(3)}$ INTO 7-CYCLES

Theorem 3.1 $K_{29}^{(3)}$ can be decomposed into 7-cycles.

Proof. By the definition of $D(n)$, researchers have $D(29)$ ={(1,1), (2,2), (3,3), (4,4), (5,5),(6,6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12), (13,13), (14,14), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (1,12), (1,13), (1, 14) , (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (2,11), (2,12), (2,13), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (3,11), (3,12), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,6), (5,7) (5,8), (5,9), (5,10), (5,11), (6,7), (6,8), (6,9), (6,10), (6,11), (7,8), (7,9), (7,10), (8,9) (8,10)}.

Now, researchers need to find the decomposition of $K_{29}^{(3)}$. On $D(29)$, according to Definition 2.5 researchers obtain 18 sequences as follows:

(1)(1,1,2,1,3,1,20) , (2)(1,4,27,5,23,28,22,) (3)(1,7,25,27,8,2,17), (4)(1,9,22,4,1,10,11), (5)(1,12,19,2,9,2,13), (6)(1,13,18,3,2,12,9) , (7)(1,14,18,4,4,5,12), (8)(1,18,13,2,23,23,7), (9)(2,15,17,25,10,12,6), (10)(2,20,12,25,12,11,5) (11)(3,3,5,19,3,9,16), (12)(3,6,3,18,5,10,13), (13)(3,14,19,6,13,22,10), (14)(3,15,18,24,12,10,5), (15)(4,13,21,22,13,7,7), (16)(4,14,8,9,10,8,5), (17)(4,15,23,15,8,13,9), (18)(5,9,11,7,5,15,6).

Let Γ is a collection of the 18 sequences above. The following 18 base 7-cycles are induced by the 18 sequences:

C7(1)=(0,1,2,4,5,8,9), C7(2)=(0,1,5,3,8,2,7), C7(3)=(0,1,8,4,2,10,12), C7(4)=(0,1,,10,3,7,8,18), C7(5)=(0,1,13,3,5,14,16), C7(6)=(0,1,14,3,6,8,20), C7(7)=(0,1,15,4,8,12,17), C7(8)=(0,1,19,3,5,28,22) , C7(9)= (0,2,17,5,1,11,23), C7(10)= (0,2,22,5,1,13,24), C7(11)=(0,3,6,11,1,4,13), C7(12)=(0,3,9,12,1,6,16), C7(13)= (0,3,17,7,13,26,19), C7(14)= (0,3,18,7,2,14,24), C7(15)=(0,4,17,9,2,15,22), C7(16)=(0,4,18,26,6,16,24), C7(17)=(0,4,19,13,28,7,20), C7(18)=(0,5,14,25,3,8,23).

By the method of edge-partition, researchers obtain the decomposition of $K_{29}^{(3)}$ into 522 7-cycles, that is:

$$\varepsilon(K_{29}^{(3)}) = \bigcup_{(k_1,k_2)\in(D(29)} H(k_1, k_2)$$
$$= \bigcup_{(k_1,k_2,\cdots,k_7)\in D_{all}(29)} H(k_1, k_2, \cdots, k_7)$$
$$= \bigcup_{i=1}^{18} \{C_{7(i)} + j, j \in \mathbf{Z}_{29}\}.$$

Hence researchers obtain the decomposition of $K_{29}^{(3)}$ into 522 7-cycles.

IV. CONCLUSIONS（$K_{29}^{(3)}$ *CAN BE DECOMPOSED INTO 7-CYCLES.*）

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REFERENCES

[1] C. Berge, Graphes et hypergraphes, Monographes universitaires de mathémathiques, Dunod, Paris, 1970.

[2] J. C. Bermond, A. Germa, M. C. Heydemann, D. Sotteau. Hypergraphes hamiltoniens, Colloq. Internat. CNRS 260, CNRS, Paris, 1978.

[3] H. Verrall, "Hamiltonian decompositions of complete 3-uniform hypergraphs". Discrete Math, 1994, vol. 32, pp: 333-348.

[4] G. Y. Katona, H. A. Kierstead, "Hamiltonian chains in hypergraphs", J. Graph Theory, 1999, vol. 30, pp: 205 -212.

[5] J. F. Wang, Tony T. Lee, "Paths and cycles of hypergraphs", Science in China (A), 1999, vol. 42, pp: 1-12.

[6] M. Meszka, A. Rosa, "Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles",. Australasian Journal of Combinatorics, 2009,vol. 45, pp: 291-302.

[7] R. A. Bailey, M. A. Ollis, D. A. Preece, "Round-dance neighbour designs from terraces", Discrete Math, 2003, vol. 266, pp: 69-86.

[8] Jirimutu, J. F. Wang, Study on graph labelings and hypergraph decomposition. Doctoral Dissertation. Dalian University of Technology, 2006.

[9] Jirimutu, J. F. Wang, "Hamiltonian decompositions of complete bipartite hypergraphs",. Acta Mathematicae Applicatae Sinica, 2001,vol. 4, pp: 563-566.

[10] J. F. Wang, G. Y. Yan, "On cycle structure of hypergraph", Chinese Science Bulletin. 2001, vol. 19, pp: 1585-1589

[11] .H. Huo, Jirimutu, Y. S. Yang, Algorithms for d ecomposing the edges of 3-uniform hypergraph in- to Hamiltonian cycles.(2015) Acta Mathematica Sinica(Chinese,in published).

[12] G. R. Li, Y. M. Lei and Jirimutu, "Decomposing the complete 3-uniform hypergraph $K_n^{(3)}$ into 5-cycles", Proc. ICREE Symp. Renewable Energy and Environmental Technology E Press, August 2014, pp. 1935-1939.

[13] G. R. Li, Y. M. Lei, Y. S. Yang and Jirimutu, "Decomposing the complete 3-uniform hypergraph $K_n^{(3)}$ into cycles", Jour.nal of Mathematical Research with Applications, 2015(in published).

[14] G. R. Li, Y. M. Lei, Y. S. Yang and Jirimutu, "Decomposing the complete 3-uniform hypergraph into 7-cycles".(Submitted)