Decomposing Complete 3-Uniform Hypergraph $K^{(3)}_n$ into 7-cycles

Hong Yan
College of Mathematics
Inner Mongolia University for the Nationalities
Tongliao, China
E-mail: nmhongyan@163.com

Jirimutu
Institute of Discrete Mathematics
Inner Mongolia University for the Nationalities
Tongliao, China
E-mail: jrmt@sina.com
* Corresponding Author

Abstract—This paper respectively introduces the Jianfang Wang-Tony Lee and Katona-Kierstead definition of a Hamiltonian cycle in a uniform hypergraph. In general, the existence of decomposition into 7-cycles remains open. In this paper, researchers use the method of edge-partition and cycle sequence proposed by Jirimutu and Jianfang Wang. Researchers find a new result.

Keywords—Uniform Hypergraph; 7-cycle; Cycle Sequence; Cycle Decomposition; Hamiltonian Cycle

I. INTRODUCTION

Hypergraphs are subsets systems of finite sets and may be regarded as one of the most general structures in discrete mathematics. As a natural generalization of Hamiltonian decomposition of graphs, researchers have the problem of Hamiltonian cycle decomposition of hypergraphs. In 1970, the notion of Hamiltonicity was first generalized to uniform hypergraphs by Berge in [1]. His definition of a Hamiltonian cycles in a hypergraph $H = (V, E)$ is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k)$, where $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $e_1, e_2, \ldots, e_n$ are distinct elements of $E$. A cycle of this type is called a Berge cycle. In 1970, the study of decomposition of complete 3-uniform hypergraphs into cycles of this type was begun by Bermond in [2]. In 1994, this problem was solved by Verrall in [3]. Many papers studied the two different definitions of a Hamiltonian cycle in [6] and [8], which are due to Katona-Kierstead Wang-Lee respectively. In fact, two different definitions of a Hamiltonian chain and Hamiltonian cycle are the same. A decomposition of the complete $K^{(3)}$-uniform hypergraphs into Hamiltonian cycles has been considered in [7] to [14]. In paper [7], Hamiltonian decompositions of $K^{(3)}_n$ for all admissible $n \leq 32$ has been resolved. Recently, by programming, using the method of edge-partition and cycle sequence, researchers have obtained some results for all admissible $32 < n \leq 46$. In [7], the problem of decomposing the complete 3-uniform hypergraph into $\lambda$. cycles ($\ell \geq 5$) are open. Meszka and Rosa have introduced a necessary condition for the existence of such a decomposition is that $n \equiv 1, 2, 5, 7, 10$ or $11 (\mod 15)$.

A decomposition of $K^{(3)}_n$ into 5-cycles exists for all admissible $n \leq 17$, for all $n = 4m + 1$, $m$ is a positive integer. In paper [13], they find a decomposition of $K^{(3)}_n$ into 5-cycles for $n \in \{5, 7, 10, 11, 16, 17, 20, 22, 26\}$ and has showed if $K^{(3)}_{5n}$ can be decomposed into 5-cycles, then $K^{(3)}_{5n}$ can be decomposed into 5-cycles. In paper [18], researchers have found a decomposition of $K^{(3)}_n$ into 7-cycles for $n \in \{7, 8, 14, 16, 22, 23\}$ and have shown if $K^{(3)}_{7n}$ can be decomposed into 7-cycles, then $K^{(3)}_{7n}$ can be decomposed into 7-cycles. In this paper, researchers find a decomposition of $K^{(3)}_{29n}$ into 7-cycles.

II. PRELIMINARIES

A hypergraph $H = (V, E)$ consists of a finite set $V$ of vertices with a family $E$ of subsets of $V$, called hyperedges (or simply edges). If each (hyper)edge has size $k$, researchers say that $H$ is a $k$-uniform hypergraph. In particular, the complete $k$-uniform hypergraph on $n$ vertices has all $k$-subsets of $\{1, 2, \ldots, n\}$ as edges, denoted this by $K^{(k)}_n$. The (hyper)edges of $K^{(3)}_n$ is denoted by $E(K^{(3)}_n)$.

Definition 2.1 Let $H = (V, E)$ be a $k$-uniform hypergraph. A $l$-cycle in $H$ is a cyclic sequence
where in paper \[10\], all difference pairs of \(v_i\), define a subhypergraph of the \(l\)-cycle decomposition of \(H\) and \(\mathcal{E}\) be a triple of \(k\)-tuples of integers, \(\mathbb{Z}\), where \(k\) be the set of integers, \(\mathbb{Z}\), a necessary and sufficient condition for the second equation is inequitable.

Clearly, the three differences sum to zero. Therefore if researchers know that the first two differences are \(x\) and \(y\), then the third is \(n - x - y\). Omit the third number, and researchers can get a difference pair. Using edge-partition of \(K_n^{(3)}\) in paper \([10]\), all difference pairs of \(K_n^{(3)}\) may be obtained. Let \(\mathbb{Z}_n\) be the set of integers, \(n\) be a fixed positive integer, and \(\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}\).

Let \(D_{\text{all}}(n) = \{(k_1, k_2) | 1 \leq k_1, k_2 \leq n - 1, \text{and } k_1 + k_2 \neq n\}\), \(D(n) = D_e \cup D_i \cup D_m\), where

\[
D_e = \{(k_1, k_2) \in D_{\text{all}}(n) | k_1 = k_2 = k, 1 \leq k < \frac{n}{2}\},
\]

\[
D_i = \{(k_1, k_2) \in D_{\text{all}}(n) | 1 \leq k_1 < k_2 < \frac{n - k_2}{2}\},
\]

\[
D_m = \{(k_1, k_2) \in D_{\text{all}}(n) | (k_1, k_2) \in D_i\}.
\]

Given a difference pair \((k_1, k_2) \in D_{\text{all}}(n)\) and an integer \(m \in \mathbb{Z}_n\), define a subhypergraph of \(K_n^{(3)}\) generated by \((k_1, k_2)\) as follows:

\[
E(m; k_1, k_2) = \{m, m+k_1, m+k_1+k_2\} (\text{mod } n),
\]

where the addition is performed modulo \(n\). Here \(m \in \mathbb{Z}_n\). When \((k_1, k_2)\) and \((k_1', k_2')\) are inequivalent when \(i \neq j\). The sequence \((k_1, k_2, \ldots, k_l)\) induces the cycle sequence \((r_0, r_1, \ldots, r_{l-1})\). This sequence satisfies the following two conditions:

\[
r_0 = 0, \quad \sum_{i=1}^{j} k_i \equiv r_j \pmod{n}, \quad r_i = 0
\]

\[
a) \quad i, j (i \neq j), \quad r_i \neq r_j.
\]

\[
b) \quad \text{For any } (r_0, r_1, \ldots, r_{l-1}) \text{ is a } l\text{-cycle, denoted by } C_l = (r_0, r_1, \ldots, r_{l-1}) \text{ and called base cycle. According to the difference pattern } \pi(T) \text{, obviously, researchers obtained the set of } l\text{-cycles } \{C_{i+1} | i \in \mathbb{Z}_n\}. \quad \text{In particular, if } l = n, (r_0, r_1, \ldots, r_{n-1}) \text{ is a base Hamiltonian cycle, denoted by } C_n = (r_0, r_1, \ldots, r_{n-1}).
\]

Theorem 2.7 Let \(n\) be a positive integer, for any \(1 \leq i, j \leq l - 1\), \((k_i, k_{i+1}) \in D_{\text{all}}(n)\) \((k_j, k_{j+1})\) and \((k_i, k_{i+1})\) are inequivalent when \(i \neq j\), obtained that
(\(k_1, k_2, \ldots, k_i\)) be a sequence on \(D_{al}(n)\). Then
\[
H(k_1, k_2, \ldots, k_j) = \bigcup_{i=1}^{j} H(k_1, k_{i+1}),
\]
where \(k_{i+1} = k_i\).

III. DECOMPOSING \(K_{29}^{(3)}\) INTO 7-CYCLES

Theorem 3.1 \(K_{29}^{(3)}\) can be decomposed into 7-cycles.

Proof. By the definition of \(D(n)\), researchers have
\[
D(29) = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9), (10,10), (11,11), (12,12), (13,13), (14,14), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (1,11), (1,12), (1,13), (1,14), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (2,11), (2,12), (2,13), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (3,11), (3,12), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,6), (5,7), (5,8), (5,9), (5,10), (5,11), (6,7), (6,8), (6,9), (6,10), (6,11), (7,8), (7,9), (7,10), (8,9), (8,10)\}.

Now, researchers need to find the decomposition of \(K_{29}^{(3)}\). On \(D(29)\), according to Definition 2.5, researchers obtain 18 sequences as follows:
\[
(1)(1,1,1,3,1,20) , (2)(1,4,27,5,23,28,22),
(3)(1,7,25,27,8,2,17), (4)(1,9,22,4,1,10,11),
(5)(1,12,19,2,9,2,13),
(6)(1,13,18,3,2,12,9) , (7)(1,14,18,4,4,5,12),
(8)(1,18,13,2,23,23,7), (9)(2,15,17,25,10,12,6),
(10)(2,20,12,5,25,12,11,5), (11)(3,3,5,19,3,9,16),
(12)(3,6,18,5,10,13), (13)(3,14,19,6,13,22,10),
(14)(3,15,18,24,12,10,5), (15)(4,13,21,22,13,7,7),
(16)(4,14,8,9,10,8,5), (17)(4,15,23,15,8,13,9),
(18)(5,9,11,7,5,15,6).

Let \(D\) is a collection of the 18 sequences above. The following 18 base 7-cycles are induced by the 18 sequences:
\[\begin{align*}
C_7(1) &= (0,1,2,4,5,8,9), & C_7(2) &= (0,1,5,3,8,2,7), & C_7(3) &= (0,1,3,8,2,7,10), & C_7(4) &= (0,1,1,2,4,5,8,9), \\
C_7(5) &= (0,1,11,3,5,14,16), & C_7(6) &= (0,1,14,3,6,8,20), & C_7(7) &= (0,1,15,4,8,12,17), & C_7(8) &= (0,1,19,5,3,28,22), & C_7(9) &= (0,2,17,5,11,12,23), \\
C_7(10) &= (0,2,22,5,13,24), & C_7(11) &= (0,3,6,11,1,4,13), & C_7(12) &= (0,3,9,12,16,16), & C_7(13) &= (0,3,17,7,13,26,19), & C_7(14) &= (0,3,18,7,12,14,24), \\
C_7(15) &= (0,4,17,9,2,15,22), & C_7(16) &= (0,4,18,26,16,24), & C_7(17) &= (0,4,19,13,28,7,20), & C_7(18) &= (0,5,14,25,3,8,23).
\end{align*}\]

By the method of edge-partition, researchers obtain the decomposition of \(K_{29}^{(3)}\) into 522 7-cycles, that is:
\[
\varepsilon(K_{29}^{(3)}) = \bigcup_{i=1}^{18} H(k_1, k_2) = \bigcup_{i=1}^{18} H(k_1, k_2, \ldots, k_7) \text{ where } k_{i+1} = k_i.
\]

Hence researchers obtain the decomposition of \(K_{29}^{(3)}\) into 522 7-cycles.

IV. CONCLUSIONS (\(K_{29}^{(3)}\) CAN BE DECOMPOSED INTO 7-CYCLES.)

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