Existence Results for Impulsive Neutral Functional Differential Equations with Delay Argument

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Abstract—By using the fixed point theorems and the decomposition and synthesis technique, the existence results for impulsive neutral functional differential equations with delay argument are established. An example to illustrate the main results is given.

Keywords—Existence Result; Impulsive Differential equation; Fixed Point; Decomposition and Synthesis technique

I. INTRODUCTION

In this paper, the authors study the existence of mild solutions for impulsive neutral functional differential equations

$$\begin{cases} (u(\phi(t)))' = f(t, u(t), u(t-\tau)), \\ t \in J = [t_0, \infty), t \neq \xi_k, k \in N, \\ \Delta u(t_k) = I_k(u(t_k)), k \in N, \\ u(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \quad (1.1)$$

where

$$t_0 < t_1 < \dots < t_k \rightarrow \infty \text{ as } k \rightarrow \infty, I_k \in C(\mathbf{R}, \mathbf{R}).$$

$\phi \in C^1(J, \mathbf{R})$, ϕ is monotone increasing with

$$t \leq \phi(t) \ (t \in J), \ \phi(t_0) = t_0, \ \phi^{-1} \in C^1(J, \mathbf{R}),$$

and

$$\phi(\xi_k) = t_k, \ 0 < \tau \leq \xi_{k+1} - \xi_k < \infty \ (k \in N)$$

$f: J \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is continuous everywhere except for some $\{\xi_k\} \times \mathbf{R}^2$ at which $f(\xi_k^+, x, y)$ and $f(\xi_k^-, x, y)$

exist,

$$f(\xi_k^-, x, y) = f(\xi_k^+, x, y).$$

Denote by $PC([t_0 - \tau, \infty), \mathbf{R})$, the set of all functions

$$u: [t_0 - \tau, \infty) \rightarrow \mathbf{R}$$

which are piecewise continuous in $[t_0 - \tau, \infty)$ with points of discontinuity of the first kind at the points $t_k \in X$, i.e., there exist the limits

$$u(t_k^+) < \infty$$

and

$$u(t_k^-) = u(t) < \infty,$$

$$\Delta u(t_k) = u(t_k^+) - u(t_k^-) \ (k \in N).$$

Impulsive differential equations have become more important for mathematical modeling of many real phenomena, especially in control, biological or medical domains (see, for example, [1-5]). Recently, several works have been devoted to the study of global existence results for impulsive differential equations. For instance, [7-9] are the ordinary differential equations and [10, 11] are the abstract differential systems. In particular, Luo and Shen [7, 8] considered the global existence results for impulsive differential equations by using the Schaefer fixed point theorem and the piecewise continuous Lyapunov-like functions. Motivated by the results of [7, 8], this paper is concerned with the existence of mild solutions for impulsive neutral functional differential equations. The main features of this paper are as follows: Firstly, to the knowledge, no paper has considered the existence of mild solutions for impulsive neutral functional differential equations. Secondly, the main tools used in this paper are

the Leray-Schauder fixed theorem, the Banach fixed theorem and the decomposition and synthesis technique, where the piecewise continuous Lyapunov like functions used in [7, 8] are not used.

Definition 1. A function $u: [t_0 - \tau, \infty) \rightarrow \mathbf{R}$ is said to be a solution of (1.1) if for any

$$a > t_0 \ (a \neq \xi_k),$$

the function u is a solution of (1.2), i.e.,

$$\begin{cases} (u(\phi(t)))' = f(t, u(t), u(t-\tau)), \\ t \in J_a = [t_0, a], \ t \neq \xi_k, \ k \in N_a, \\ \Delta u(t_k) = I_k(u(t_k)), \ k \in N_a, \\ u(t) = \varphi(t), \ t \in [t_0 - \tau, t_0], \end{cases} \quad (1.2)$$

where

$$N_a = \{1, \dots, n_a \mid n_a = \max_{\xi_k < a} k\},$$

and let

$$J_a^* = J_a \setminus \{t_1, \dots, t_{n_a}\}.$$

II. PRELIMINARIES

Let $E_{a^0} = \{u \mid u \in PC(J_a, \mathbf{R})\} \cap C^1(J_a^*, \mathbf{R})$.

Evidently, E_{a^0} is a real Banach space with norm

$$\|u(t)\|_{a^0} = \sup_{t \in J_a} |u(t)|.$$

Further, let

$$E_a = \{u(\phi(t)) \mid u(t) \in E_{a^0}\}.$$

We can check that E_a is also a real Banach space with norm

$$\|u(t)\|_a = \sup_{t \in J_a} |u(\phi(t))|.$$

According to the properties of ϕ . The authors have

$$\begin{aligned} \|u(t)\|_{a^0} &= \sup_{t \in J_a} |u(t)| \\ &\leq \sup_{t \in J_a} |u(\phi(t))| = \|u(t)\|_a. \end{aligned} \quad (2.1)$$

Define operator

$$B: u(t) \mapsto u(\phi(t)),$$

where

$$u(t) \in E_{a^0} \text{ and } u(\phi(t)) \in E_a.$$

It is evident that B is topological linear isomorphic, which implies that E_a is a real Banach space.

Lemma 1. $u(t) \in E_{a^0}$ is a solution of (1.2) if and only if $u(t) \in E_{a^0}$ is a solution of the following integral equation

$$\begin{aligned} u(\phi(t)) &= u_0 + \int_{t_0}^t f(s, u(s), u(s-\tau)) ds + \\ &\quad \sum_{t_0 < \xi_k < t} I_k(u(t_k)), \quad t \in J_a, \end{aligned} \quad (2.2)$$

where

$$u(t) = \varphi(t), t \in [t_0 - \tau, t_0].$$

Since this proof is similar to the one of Lemma 2 in [12], the authors omit it.

Lemma 2 (Leray-Schauder [5]). Let operator

$$A: X \rightarrow X$$

be completely continuous, where X is a real Banach space. If the set

$$G = \{\|x\| \mid x \in X, x = \lambda Ax, 0 < \lambda < 1\}$$

is bounded, then the operator A has at least one fixed point in the closed ball

$$T = \{x \mid x \in X, \|x\| \leq R\},$$

where

$$R = \sup G.$$

Lemma 3 (Compactness criterion [6]). $H \subset PC(J, \mathbf{R})$

is a relatively compact set if and only if

$$H \subset PC(J, \mathbf{R})$$

is uniformly bounded and equicontinuous on every

$$A_k \ (k = 0, 1, \dots),$$

where

$$A_0 = [t_0, t_1]; \ A_k = (t_k, t_{k+1}] \ (k = 1, 2, \dots).$$

III. MAIN RESULT

Let us introduce the following conditions for later use:

(H1) There for any $a > t_0$ ($a \neq \xi_k$) exist non-negative constants

$$b, \ c_k \ (k \in N), \text{ and } g_1 \in L(J, \mathbf{R}^+)$$

such that

$$\begin{aligned} &|f(t, u_2(t), u_2(t-\tau)) - f(t, u_1(t), u_1(t-\tau))| \\ &\leq b g_1(t) \|u_2 - u_1\|_{a^0}, \ t \in J_a, \\ &|I_k(x_2(t_k)) - I_k(x_1(t_k))| \leq c_k |x_2(t_k) - x_1(t_k)|, \\ &I_k(0) = 0, \ k \in N_a, \end{aligned}$$

where

$$u_1, u_2 \in E_{a^0}$$

with

$$u_1(t) = u_2(t) = \varphi(t) \ (t \in [t_0 - \tau, t_0]),$$

$$c = \sum_{k=1}^{\infty} c_k, \ d_1 = \int_{t_0}^{\infty} g_1(t) dt.$$

(H2) There exist $g_2 \in L(J, \mathbf{R}^+) \cap PC(J, \mathbf{R}^+)$ such that

$$|f(t, u(t), u(t-\tau))| \leq g_2(t) (1 + \|u(t)\|_{a^0}), \quad t \in J_a,$$

where

$$d_2 = \int_{t_0}^{\infty} g_2(t) dt.$$

(H3) $l = c + d_2 < 1$.

(H4) $r = bd_1 + c < 1$.

Theorem 1. If conditions (H1), (H2) and (H3) are satisfied, then (1.1) has at least one solution.

Proof. (i) For any

$$a > t_0 \quad (a \neq \xi_k) \text{ and } u(\phi(t)) \in E_a,$$

define operator A by

$$\begin{aligned} Au(\phi(t)) = & u_0 + \int_{t_0}^t f(s, u(s), u(s-\tau)) ds + \\ & \sum_{t_0 < \xi_k < t} I_k(u(t_k)), \quad t \in J_a. \end{aligned} \quad (3.1)$$

It is easy to see that

$$Au(\phi(t)) \in E_{a^0}.$$

According to the properties of ϕ , for any

$$v(t) \in E_{a^0},$$

the authors have

$$v(t) = v(\phi^{-1}(\phi(t))) = v\phi^{-1}(\phi(t)).$$

Let $u = v\phi^{-1}$. Next, it is clear that

$$v(t) = u(\phi(t)) \in E_a.$$

It follows that A maps E_a into E_a . Thus

$$Au(\phi(t)) \in E_a.$$

A is a completely continuous operator will be verified by the following three steps.

Step1. A is continuous.

Let any $u_n(\phi(t))$ ($n \in N$), $u(\phi(t)) \in E_a$

with

$$\|u_n(\phi(t)) - u(\phi(t))\|_a \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By (3.1), (H1) and (2.1), the authors have

$$\begin{aligned} & |Au_n(\phi(t)) - Au(\phi(t))| \\ & \leq b \int_{t_0}^t g_1(s) \|u_n(s) - u(s)\|_{a^0} ds + \sum_{t_0 < \xi_k < t} c_k |u_n(t_k) - u(t_k)| \\ & \leq b \|u_n(t) - u(t)\|_{a^0} \int_{t_0}^t g_1(s) ds + \|u_n(t) - u(t)\|_{a^0} \sum_{t_0 < \xi_k < t} c_k \\ & \leq b \|u_n(t) - u(t)\|_{a^0} \int_{t_0}^{\infty} g_1(s) ds + \|u_n(t) - u(t)\|_{a^0} \sum_{k=1}^{\infty} c_k \end{aligned}$$

$$\leq b \|u_n(t) - u(t)\|_{a^0} \int_{t_0}^{\infty} g_1(s) ds + \|u_n(t) - u(t)\|_{a^0} \sum_{k=1}^{\infty} c_k$$

Then

$$\begin{aligned} & \|Au_n(\phi(t)) - Au(\phi(t))\|_a \\ & \leq (bd_1 + c) \|u_n(\phi(t)) - u(\phi(t))\|_a. \end{aligned} \quad (3.2)$$

Thus

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_a \rightarrow 0 \text{ as } n \rightarrow \infty,$$

that is to say, A is continuous.

Step 2. A maps any bounded subset of E_a into one bounded subset of E_a .

Let T be any bounded subset of E_a . Then there exist $h > 0$ such that

$$\|u(\phi(t))\| \leq h \text{ for all } u(\phi(t)) \in T.$$

By (3.1), (H1), (H2) and (2.1), the authors have

$$\begin{aligned} & |Au(\phi(t))| \\ & \leq |u_0| + \int_{t_0}^t g_2(s) (1 + \|u(s)\|_{a^0}) ds + \sum_{t_0 < \xi_k < t} c_k |u(t_k)| \\ & \leq |u_0| + (1 + \|u(t)\|_{a^0}) \int_{t_0}^t g_2(s) ds + \|u(t)\|_{a^0} \sum_{t_0 < \xi_k < t} c_k \\ & \leq |u_0| + d_2 (1 + \|u(\phi(t))\|_a) + c \|u(\phi(t))\|_a \\ & \leq |u_0| + d_2(1 + h) + ch, \quad t \in J_a, \end{aligned}$$

So

$$\|Au(\phi(t))\|_a \leq |u_0| + d_2(1 + h) + ch. \quad (3.3)$$

Therefore $A(T)$ is uniformly bounded.

Step 3. $A(T)$ is equicontinuous on every $\bar{J}_k$ ($k = 0, \dots, n_a$), where

$$\bar{J}_0 = [t_0, \xi_1], \quad \bar{J}_k = (\xi_k, \xi_{k+1}] \quad (k = 1, \dots, n_a - 1)$$

and $\bar{J}_{n_a} = (\xi_{n_a}, a]$.

For any $Au(\phi(t)) \in A(T)$ and $\varepsilon > 0$, take

$$\delta = [(1 + \|u(t)\|_{a^0}) \max_{t \in J_a} g_2(t)]^{-1} \varepsilon.$$

Then if $t_1, t_2 \in \bar{J}_k$ and $|t_1 - t_2| < \delta$ with $t_1 < t_2$, from (3.1) and (H2), the authors have

$$\begin{aligned} & |Au(\phi(t_2)) - Au(\phi(t_1))| \leq \int_{t_1}^{t_2} g_2(s) (1 + \|u(s)\|_{a^0}) ds \\ & \leq (1 + \|u(t)\|_{a^0}) \max_{t \in J_a} g_2(t) (t_2 - t_1) < \varepsilon. \end{aligned}$$

Thus, $A(T)$ is equicontinuous on every $\bar{J}_k$ ($k = 0, \dots, n_a$).

As a consequence of Step 1-3, A is completely continuous.

(ii) For any

$$\begin{aligned} \|u(\phi(t))\|_a &\in G = \{\|u(\phi(t))\|_a \mid u(\phi(t)) \in E_a, \\ u(\phi(t)) &= \lambda Au(\phi(t)), 0 < \lambda < 1\}, \end{aligned}$$

similar with getting (3.3), the authors have

$$\begin{aligned} \|u(\phi(t))\|_a &= \lambda \|Au(\phi(t))\|_a \leq \|Au(\phi(t))\|_a \\ &\leq |u_0| + d_2 + (c + d_2) \|u(\theta(t))\|_a \\ &= L + l \|u(\phi(t))\|_a, \end{aligned}$$

where

$$L = |u_0| + d_2.$$

It follows that

$$\|u(\phi(t))\|_a \leq \frac{L}{1-l},$$

i.e., G is bounded.

From (i) and (ii), now all conditions of Lemma 2 are satisfied, so (1.2) has at least one solution for any $a > t_0$ ($a \neq \xi_k$). Let $u_a(t)$ be one solution of (1.2). It is clear that $u_{a_2}|_{J_{a_1}}$ is one solution of (1.2) in J_{a_1} for any $a_2 > a_1 > t_0$. Now, the authors get

$$u_k(t) = u_{k+1}|_{J_k} \quad (k \in N).$$

Then let $u(t) = u_k(t)$, $t \in [t_0, k]$ ($k \in N$), and

$$u(t) = \varphi(t), t \in [t_0 - \tau, t_0].$$

By Definition 1, $u(t)$ is one solution of (1.1). Therefore the proof is complete.

Theorem 2. If conditions (H1) ($I_k(0) = 0$ are not needed) and (H4) are satisfied, then (1.1) has a unique solution.

Proof. Similar with getting (3.2), for any

$$a > t_0 \quad (a \neq \xi_k) \text{ and } u_1(\phi(t)), u_2(\phi(t)) \in E_a,$$

we obtain

$$\begin{aligned} &\|Au_2(\phi(t)) - Au_1(\phi(t))\|_a \\ &\leq [(b + k_0 b_1 + h_0 b_2) d_1 + c] \|u_2(\phi(t)) - u_1(\phi(t))\|_a \\ &= \gamma \|u_2(\phi(t)) - u_1(\phi(t))\|_a. \end{aligned}$$

Thus all conditions of the fixed point theorem of Banach are satisfied.

So (1.2) has a unique solution for any

$$a > t_0 \quad (a \neq \xi_k).$$

The authors take similarly $u(t)$ with Theorem 1. By Definition 1, $u(t)$ is a unique solution of (1.1). The proof is complete.

IV. An EXAMPLE

Consider the equation

$$\begin{cases} (u(3t))' = \frac{e^{-t}}{6} [u(t) + u(t-1)], \\ t \in J = [0, \infty), t \neq \xi_k = k \quad (k \in N), \\ \Delta u(t_k) = \frac{1}{3^{k+1}} u(t_k), t_k = 3k \quad (k \in N), \\ u(t) = e^t, \quad t \in [-1, 0], \end{cases} \quad (4.1)$$

Firstly, $\phi(t) = 3t$, and the requisitions of (1.1) are satisfied.

Next, since

$$f(t, x, y) = \frac{e^{-t}}{6} (x + y),$$

for any

$$a > t_0 \quad (a \neq \xi_k = k),$$

the authors have

$$\begin{aligned} &|f(t, u_2(t), u_2(t-1)) - f(t, u_1(t), u_1(t-1))| \\ &\leq \frac{e^{-t}}{6} [|u_2(t) - u_1(t)| + |u_2(t-1) - u_1(t-1)|] \\ &\leq \frac{e^{-t}}{3} \|u_2 - u_1\|_{a^0}, t \in J_a, \end{aligned}$$

where

$$b = \frac{1}{3}, \quad g_1(t) = e^{-t}, \quad d_1 = \int_0^\infty e^{-t} dt = 1.$$

From

$$I_k(x) = \frac{1}{3^{k+1}} x,$$

we have

$$|I_k(x_2(t_k)) - I_k(x_1(t_k))| \leq \frac{1}{3^{k+1}} |x_2(t_k) - x_1(t_k)|,$$

$$I_k(0) = 0,$$

where

$$c_k = \frac{1}{3^{k+1}}, \quad c = \sum_{k=1}^\infty \frac{1}{3^{k+1}} = \frac{1}{6}.$$

Further, for any

$$a > t_0 \quad (a \neq \xi_k),$$

we have

$$|f(t, u(t), u(t-1))| \leq \frac{e^{-t}}{6} [|u(t)| + |u(t-1)|]$$

$$\leq \frac{e^{-t}}{3} [1 + \|u(t)\|_{a^0}], \quad t \in J_a,$$

where

$$g_2(t) = \frac{e^{-t}}{3}, \quad d_2 = \int_0^\infty \frac{e^{-t}}{3} dt = \frac{1}{3}.$$

Finally, the authors get

$$l = c + d_2 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} < 1.$$

Therefore, (4.1) satisfies all conditions of Theorem 1. It follows that (4.1) has at least one solution.

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