Multidimensional Discrete Green's Function and Its Estimates

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Abstract—This paper will discuss estimates for discrete Green's function of elliptic equations in dimensions seven and up. First, the definitions of some terms are given. Then the estimates for the regularized Green's function are derived. Finally, by using the triangular inequality, researchers obtain the estimates for discrete Green's function. The results of the paper play important roles in the research of superconvergence of finite element methods.

Keywords—Regularized Green's Function; Discrete Green's Function; Superconvergence; Finite Element; Elliptic Equations

I. INTRODUCTION

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1-8]). For one- and two-dimensional elliptic problems, one has obtained many optimal estimates for the Green's function (see [1]). Recently, to solve the three-dimensional elliptic problems, the $W^{2,1}$ -seminorm optimal estimate with order $O(|\ln h|^{\frac{2}{3}})$ for the discrete Green's function and the $W^{1,1}$ -seminorm optimal estimate with order $O(|\ln h|^{\frac{4}{3}})$ for the discrete derivative Green's function were derived (see [5, 9]). It might seem that in current practical applications of the finite element method there is no need for simplicial higher-dimensional elements. Nevertheless, it is well known that for example in areas like financial mathematics, particle physics, statistical physics and general relativity, higher-dimensional pdes need to be solved (see [10]).

Therefore, it is meaningful to study the estimates for the Green's function in four and up space dimensions. Recently, for dimensions four to six, researchers have also considered the estimates for discrete Green's functions. In this paper, researchers will derive some estimates for the discrete Green's function for dimensions seven and up. Researchers will find that there are not the optimal estimates for discrete Green's functions other than those in dimensions one to six.

In this paper, researchers shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

II. BOUNDARY VALUE PROBLEM AND FINITE ELEMENTS DISCRETIZATION

The model problem that researchers study in this paper is

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (1)$$

Where $\Omega \subset R^d$ ($d \geq 7$) is a bounded polytopic domain. Let $\{T^h\}$ be a regular family of uniform partitions of $\bar{\Omega}$, and $S_0^h(\Omega) \subset H_0^1(\Omega)$ the finite element space.

Discretizing the above elliptic equation using $S_0^h(\Omega)$ as approximating space means finding $u_h \in S_0^h(\Omega)$ such that $a(u_h, v) = (f, v)$ for all $v \in S_0^h(\Omega)$, where

$$a(u_h, v) \equiv \int_{\Omega} \nabla u_h \cdot \nabla v \, dX$$

$$(f, v) \equiv \int_{\Omega} f v \, dX.$$

In addition, researchers also assume that the given function f is smooth enough. For every $Z \in \Omega$, researchers define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, and the L^2 -projection $P_h u \in S_0^h(\Omega)$ such that

$$(v, \delta_Z^h) = v(Z) \quad \forall v \in S_0^h(\Omega), \quad (2)$$

$$(u - P_h u, v) = 0 \quad \forall v \in S_0^h(\Omega). \quad (3)$$

Let $G_Z^* \in H^2(\Omega) \subset H_0^1(\Omega)$ be the solution of the elliptic problem $-\Delta G_Z^* = \delta_Z^h$. Researchers may call G_Z^* the regularized Green's function. Further, let the discrete

Green's function $G_Z^h \in S_0^h(\Omega)$ be the finite element approximation to G_Z^* . Thus,

$$a(G_Z^* - G_Z^h, v) = 0 \quad \forall v \in S_0^h(\Omega).$$

The purpose of this paper is to bind the term $|G_Z^h|_{2,1,\Omega}^h$.

The rest of this paper is organized as follows. In Section 3 researchers derive the $W^{2,1}$ -seminorm estimate for G_Z^* . Section 4 is devoted to the estimate for G_Z^h .

III. ESTIMATES FOR THE REGULARIZED GREEN'S FUNCTION G_Z^*

We first introduce the weight function defined by

$$\phi \equiv \phi(X) = (|X - \bar{X}|^2 + \theta^2)^{-\frac{d}{2}} \quad \forall X \in \bar{\Omega},$$

Where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in (d, +\infty)$ is a suitable real number.

For every $\alpha \in R$, researchers give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} |D^\beta v|^2, |\nabla^n v|_{\phi^\alpha, \Omega} = \left(\int_{\Omega} \phi^\alpha |\nabla^n v|^2 dX \right)^{\frac{1}{2}},$$

$$\|v\|_{m, \phi^\alpha, \Omega}^2 = \sum_{n=0}^m |\nabla^n v|_{\phi^\alpha, \Omega}^2,$$

Where $\beta = (\beta_1, \dots, \beta_d)$, $|\beta| = \beta_1 + \dots + \beta_d$, and $\beta_i \geq 0, i = 1, \dots, d$ are integers. In particular, for the case of $m = 0$, researchers write

$$\|v\|_{\phi^\alpha, \Omega} = \left(\int_{\Omega} \phi^\alpha |v|^2 dX \right)^{\frac{1}{2}}.$$

We assume that there exists a $q_0 (1 < q_0 \leq +\infty)$ such that

$$-\Delta : W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \longrightarrow L^q(\Omega) \quad (1 < q < q_0)$$

That is (the so-called a priori estimate)

$$\|v\|_{2,q,\Omega} \leq C(q) \|\Delta v\|_{0,q,\Omega}, \quad (5)$$

Where $C(q)$ denotes a constant depending on q . The weight function ϕ possesses the following properties:

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n) \phi^{\alpha + \frac{n}{2}}, \quad \alpha \in R, n = 1, 2. \quad (6)$$

$$\int_{\Omega} \phi^\alpha dX \leq C(\alpha - 1)^{-1} \theta^{-d(\alpha-1)} \quad \forall \alpha > 1, \quad (7)$$

$$\int_{\Omega} \phi dX \leq C(k) |\ln \theta|, \quad \theta \leq k < 1. \quad (8)$$

In addition, researchers have for the L^2 -projection operator

$$\|P_h w\|_{1,q,\Omega} \leq C(q) \|w\|_{1,q,\Omega} \quad \text{for } 1 \leq q \leq \infty. \quad (9)$$

In the following, researchers will give some estimates for the regularized Green's function.

Lemma 1. For δ_Z^h the discrete δ function defined by (2), researchers have the weighted-norm estimate

$$\|\delta_Z^h\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)}{2}} \quad \forall \alpha > 0. \quad (10)$$

Proof. Obviously,

$$\begin{aligned} \|\delta_Z^h\|_{\phi^{-\alpha}}^2 &\leq C \int_{\Omega} (|X - Z|^2 + \theta^2)^{\frac{d\alpha}{2}} h^{-2d} e^{-Ch^{-1}|X-Z|} dX \\ &\leq C \int_0^\infty (r^2 + \theta^2)^{\frac{d\alpha}{2}} h^{-2d} e^{-Ch^{-1}r} r^{d-1} dr. \end{aligned}$$

Set $h^{-1}r = t$, thus

$$\begin{aligned} \|\delta_Z^h\|_{\phi^{-\alpha}}^2 &\leq Ch^{(\alpha-1)d} \int_0^\infty (t^2 + \gamma^2)^{\frac{d\alpha}{2}} e^{-Ct} t^{d-1} dt \\ &\leq Ch^{(\alpha-1)d}. \end{aligned}$$

Further,

$$\|\delta_Z^h\|_{\phi^{-\alpha}}^2 \leq Ch^{d(\alpha-1)}$$

Which is the result (10).

Lemma 2. For $q_0 > 2$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$,

There exists a constant $C = C(\alpha, \Omega) > 0$ such that

$$\|\nabla^2 v\|_{\phi^{-\alpha}}^2 \leq C \left(\|\Delta v\|_{\phi^{-\alpha}}^2 + |v|_{1, \phi^{-\alpha+\frac{3}{2}}}^2 + \|v\|_{\phi^{-\alpha+\frac{3}{2}}}^2 \right) \quad \forall \alpha \in R. \quad (11)$$

Proof.

$$\begin{aligned} \|\nabla^2 v\|_{\phi^{-\alpha}}^2 &= \int_{\Omega} \phi^{-\alpha} |\nabla^2 v|^2 dx = \int_{\Omega} (\phi^{-\frac{\alpha}{2}} |\nabla^2 v|)^2 dx \\ &\leq C \left(\int_{\Omega} |\nabla^2 (\phi^{-\frac{\alpha}{2}} v)|^2 dx + \int_{\Omega} |v \nabla^2 \phi^{-\frac{\alpha}{2}}|^2 dx + \int_{\Omega} |\nabla \phi^{-\frac{\alpha}{2}}|^2 |\nabla v|^2 dx \right) \\ &\leq C \left(\|\nabla^2 (\phi^{-\frac{\alpha}{2}} v)\|_0^2 + \|v\|_{\phi^{-\alpha+\frac{1}{2}}}^2 + |v|_{1, \phi^{-\alpha+\frac{3}{2}}}^2 \right) \\ &\leq C \left(\|\Delta (\phi^{-\frac{\alpha}{2}} v)\|_0^2 + \|v\|_{\phi^{-\alpha+\frac{1}{2}}}^2 + |v|_{1, \phi^{-\alpha+\frac{3}{2}}}^2 \right) \\ &\leq C \left(\int_{\Omega} \phi^{-\alpha} |\Delta v|^2 dx + \int_{\Omega} |\nabla \phi^{-\frac{\alpha}{2}}|^2 |\nabla v|^2 dx \right. \\ &\quad \left. + \int_{\Omega} |\Delta \phi^{-\frac{\alpha}{2}}|^2 |v|^2 dx + \|v\|_{\phi^{-\alpha+\frac{1}{2}}}^2 + |v|_{1, \phi^{-\alpha+\frac{3}{2}}}^2 \right) \\ &\leq C \left(\|\Delta v\|_{\phi^{-\alpha}}^2 + |v|_{1, \phi^{-\alpha+\frac{3}{2}}}^2 + \|v\|_{\phi^{-\alpha+\frac{1}{2}}}^2 \right). \end{aligned}$$

Lemma 3. Suppose $q_0 > 2$ and $\alpha \in R$. For G_Z^* and δ_Z^h , the regularized Green's function and the discrete δ function, respectively, researchers have

$$\|\nabla^2 G_Z^*\|_{\phi^{-\alpha}}^2 \leq C \|\delta_Z^h\|_{\phi^{-\alpha}}^2 + C \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}}^2. \quad (12)$$

Proof. Taking $v = G_Z^*$ in (11), and by the definition of G_Z^* , researchers have

$$\begin{aligned} \|\nabla^2 G_Z^*\|_{\phi^{-\alpha}}^2 &\leq C \left(\|\Delta G_Z^*\|_{\phi^{-\alpha}}^2 + |G_Z^*|_{1, \phi^{-\alpha+\frac{3}{2}}}^2 + \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}}^2 \right) \\ &\leq C \|\delta_Z^h\|_{\phi^{-\alpha}}^2 + C \left| a \left(G_Z^*, \phi^{-\alpha+\frac{3}{2}} G_Z^* \right) \right| + C \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}}^2 \\ &\leq C \|\delta_Z^h\|_{\phi^{-\alpha}}^2 + C \left| \left(\delta_Z^h, \phi^{-\alpha+\frac{3}{2}} G_Z^* \right) \right| + C \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}}^2 \\ &\leq C \|\delta_Z^h\|_{\phi^{-\alpha}}^2 + C \|\delta_Z^h\|_{\phi^{-\alpha}} \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}} + C \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}}^2 \\ &\leq C \|\delta_Z^h\|_{\phi^{-\alpha}}^2 + C \|G_Z^*\|_{\phi^{-\alpha+\frac{1}{2}}}^2, \end{aligned}$$

Which completes the proof of the result (12).

Lemma 4. Suppose $q_0 > 2$ and α satisfying $\frac{4}{d}-1 < \alpha < \frac{4}{d}-1 + \min\{1, \frac{d}{2}-1\}$. Then for G_Z^* and δ_Z^h , the regularized Green's function and the discrete δ function, respectively, researchers have

$$\|\nabla^2 G_Z^*\|_{\phi^{-\alpha}}^2 \leq Ch^{d(\alpha-1)}. \quad (13)$$

Theorem 1. Suppose $q_0 > 2$ and $d \geq 7$. For G_Z^* , the regularized Green's function, researchers have

$$\|G_Z^*\|_{2,1} \leq Ch^{2-\frac{d}{2}}. \quad (14)$$

Proof. Obviously,

$$\|G_Z^*\|_{2,1}^2 \leq \int_{\Omega} \phi^\alpha dX \cdot \|\nabla^2 G_Z^*\|_{\phi^{-\alpha}}^2.$$

It is easy to prove

$$\int_{\Omega} \phi^\alpha dX \leq C(1-\alpha)^{-1} \forall 0 < \alpha < 1$$

Thus when $\frac{4}{d}-1 < \alpha < \frac{4}{d}-1 + \min\{1, \frac{d}{2}-1\}$, from (13),

$$\|G_Z^*\|_{2,1}^2 \leq C \frac{h^{d(\alpha-1)}}{1-\alpha}.$$

Hence researchers have

$$\|G_Z^*\|_{2,1}^2 \leq C \inf_{\frac{4}{d}-1 < \alpha < \frac{4}{d}} \frac{h^{d(\alpha-1)}}{1-\alpha} = Ch^{4-d}.$$

The result (14) is obtained.

IV. ESTIMATES FOR THE DISCRETE GREEN'S FUNCTION

$$G_Z^h$$

Lemma 4. Suppose $q_0 > 2$, researchers have

$$\|G_Z^* - G_Z^h\|_{1,\phi^{-\alpha}}^2 \leq Ch^{2+d(\alpha-1)}.$$

Where

$$\max\left\{\frac{4}{d}-1, 0\right\} < \alpha < \min\left\{\frac{4}{d}, \frac{4}{d} + \frac{d}{2} - 2, 1 - \frac{2}{q_0} + \frac{2}{d}\right\}. \quad (16)$$

Proof.

$$\begin{aligned} & \|G_Z^* - G_Z^h\|_{1,\phi^{-\alpha}}^2 \\ & \leq Ch^2 \|\nabla^2(G_Z^*)\|_{\phi^{-\alpha}}^2 + C \|G_Z^* - G_Z^h\|_{\phi^{-\alpha+\frac{2}{d}}}^2 \\ & \leq \hat{C} \left(h^2 \|\nabla^2(G_Z^*)\|_{\phi^{-\alpha}}^2 + \|G_Z^* - G_Z^h\|_{\phi^{-\alpha+\frac{2}{d}}}^2 \right). \end{aligned}$$

In addition, researchers get

$$\|G_Z^* - G_Z^h\|_{\phi^{-\alpha+\frac{2}{d}}}^2 \leq \frac{2}{3\hat{C}} \|G_Z^* - G_Z^h\|_{1,\phi^{-\alpha}}^2.$$

Then researchers have

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$$\|G_Z^* - G_Z^h\|_{1,\phi^{-\alpha}}^2 \leq Ch^2 \|\nabla^2(G_Z^*)\|_{\phi^{-\alpha}}^2.$$

From (13)

$$\|\nabla^2(G_Z^*)\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)}{2}}.$$

By the above inequalities, researchers immediately obtain the result (15).

Lemma 5. Suppose $d \geq 7$ and $q_0 \geq \frac{2d}{d-2}$. For G_Z^* and G_Z^h , the regularized Green's function and the discrete Green's function, respectively, researchers have

$$\|G_Z^* - G_Z^h\|_{1,1} \leq Ch^{3-\frac{d}{2}}. \quad (17)$$

Proof. Obviously,

$$\|G_Z^* - G_Z^h\|_{1,1}^2 \leq \int_{\Omega} \phi^\alpha dX \cdot \|G_Z^* - G_Z^h\|_{1,\phi^{-\alpha}}^2.$$

Since

$$\int_{\Omega} \phi^\alpha dX \leq C(1-\alpha)^{-1} \forall 0 < \alpha < 1$$

When $d \geq 7$ and $q_0 \geq \frac{2d}{d-2}$, from (16) researchers have $0 < \alpha < \frac{4}{d} < 1$. Thus from (15) researchers get

$$\|G_Z^* - G_Z^h\|_{1,1}^2 \leq C \inf_{\alpha} \frac{h^{2+d(\alpha-1)}}{1-\alpha} = Ch^{6-d}.$$

Namely

$$\|G_Z^* - G_Z^h\|_{1,1} \leq Ch^{3-\frac{d}{2}}.$$

The result (17) is obtained.

Theorem 2. Suppose $d \geq 7$ and $q_0 \geq \frac{2d}{d-2}$, researchers have

$$\|G_Z^h\|_{2,1}^h \leq Ch^{2-\frac{d}{2}}. \quad (18)$$

(15) Proof. Obviously,

$$\begin{aligned} \|G_Z^* - G_Z^h\|_{2,1}^h & \leq \|G_Z^* - \Pi G_Z^*\|_{2,1}^h + \|\Pi G_Z^* - G_Z^h\|_{2,1}^h \\ & \leq C \|G_Z^*\|_{2,1} + Ch^{-1} \|\Pi G_Z^* - G_Z^h\|_{1,1} \\ & \leq C \|G_Z^*\|_{2,1} + Ch^{-1} \|\Pi G_Z^* - G_Z^h\|_{1,1} + Ch^{-1} \|G_Z^* - G_Z^h\|_{1,1} \\ & \leq C \|G_Z^*\|_{2,1} + Ch^{-1} \|G_Z^* - G_Z^h\|_{1,1}, \end{aligned}$$

Where Π is an interpolation operator. By the triangular Inequality,

$$\|G_Z^h\|_{2,1}^h \leq \|G_Z^* - G_Z^h\|_{2,1}^h + \|G_Z^*\|_{2,1} \leq C \|G_Z^*\|_{2,1} + Ch^{-1} \|G_Z^* - G_Z^h\|_{1,1}, \quad (19)$$

From (14), (17) and (19), researchers immediately obtain the result (18).

REFERENCES

- [1] Q. D. Zhu and Q. Lin, "Superconvergence theory of the finite element methods (in Chinese)," Changsha: Hunan Science and Technology Press, 1989.
- [2] C. M. Chen and Y. Q. Huang, "High accuracy theory of finite element methods (in Chinese)," Changsha: Hunan Science and Technology Press, 1995.
- [3] C. M. Chen, "Construction theory of superconvergence of finite elements (in Chinese)," Changsha: Hunan Science and Technology Press, 2001.
- [4] G. Goodsell, "Pointwise superconvergence of the gradient for the linear tetrahedral element," *Numer. Meth. Part. Differ. Equ.*, Vol. 10, 1994, pp. 651-666.
- [5] J. H. Liu, B. Jia, and Q. D. Zhu, "An estimate for the three-dimensional discrete Green's function and applications," *J. Math. Anal. Appl.*, Vol. 370, 2010, pp. 350-363.
- [6] J. H. Liu and Q. D. Zhu, "Uniform superapproximation of the derivative of tetrahedral quadratic finite element approximation," *J. Comput. Math.*, Vol. 23, 2005, pp. 75-82.
- [7] J. H. Liu and Q. D. Zhu, "Maximum-norm superapproximation of the gradient for the trilinear block finite element," *Numer. Meth. Part. Differ. Equ.*, Vol. 23, 2007, pp. 1501-1508.
- [8] J. H. Liu and Q. D. Zhu, "Pointwise supercloseness of tensor-product block finite elements," *Numer. Meth. Part. Differ. Equ.*, Vol. 25, 2009, pp. 990-1008.
- [9] J. H. Liu and Q. D. Zhu, "The estimate for the $W_{1,1}$ -seminorm of discrete derivative Green's function in three dimensions (in Chinese)," *J. Hunan Univ. Arts Sci.*, Vol. 16, 2004, pp. 1-3.
- [10] J. H. Brandts and M. Krizek, "Gradient superconvergence on uniform simplicial partitions of polytopes," *IMA. J. Numer. Anal.*, Vol. 23, 2003, pp. 489-505.
- [11] J. H. Liu and Y. S. Jia, "Five-Dimensional Discrete Green's Function and Its Estimates," *J. Comp. Anal. Appl.*, Vol. 18, 2015, pp. 620-627.