Higher Order Crack Tip Fields for Physical Weak-Discontinuous Crack of Linear FGMs Cylindrical Shell with Reissner's Effect

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Keywords: crack tip fields, linear FGMs, eigen-expansion method, Reissner shell.

Abstract： The physical weak-discontinuous problem of an axis-directional interfacial crack between homogeneous material and functionally graded materials (FGMs) cylindrical shells is studied. Base on Reissner plate theory, the governing equations are derived for weak-discontinuous problems of cylindrical shells. The higher order crack tip fields of homogeneous materials and FGMs regions are given and assembled. The crack tip fields obtained also have the same nature of eigen-function as Williams' solution and the structure of higher order crack tip fields is analysed.

Introduction

It has been widely accepted that functionally graded materials is completely different from the homogeneous materials or the traditional composite materials. Its macro-interface is eliminated because its components and microstructure changes gradually, especially for the shell structure made of FGMs. In general, FGMs shell are of great resistance to corrosion, radiation, high temperature and other engineering applications. The higher order crack-tip fields of Reissner's FGMs plates obtained by Liu[1]. Zhang[2] studied the crack problem of Reissner's functionally graded spherical and cylindrical shells and he higher order crack-tip fields are obtained. In the engineering applications, FGMs mainly appear in the coating and interface layer, forming lots of interface structures. Due to the feature of production technology, there are a number of interface defects in structures. Furthermore, due to differences in material performance across interfaces(i.e. the differentials of material parameters at interfaces are different, which is defined the physical weak-discontinuous), the interfacial fracture becomes the main form of structure failure. Consequently, the study on weak-discontinuous fracture of the FGM cylindrical shell is of great significance. In this paper, the physical weak-discontinuous problem of an interfacial crack is considered. Assume that the area below the interface is homogeneous materials region, and the one above the interface is FGMs region. The gradient direction is along y-axis. In this paper, we extend the Williams' solution to physical weak-discontinuous problem of an interfacial crack between homogeneous material and FGMs cylindrical shells and the higher order fields are obtained.

Higher order crack tip field

The elastic modulus function form of FGMs is assumed to be

$$E^{(\mathrm{II})} = E^{(\mathrm{I})}(1+g\,y) = E^{(\mathrm{I})}(1+g\,r\sin q) \tag{1}$$

where, $E^{(\mathrm{I})}$ is the elastic modulus of homogeneous material, and $g \geq 0$ are the non-homogeneity parameter. The variation of Poisson's ratios has very insignificant effect on the stress intensity factor of non-homogeneous materials[3]. So, Poisson's ratio is assumed to be the constant m .

Assumed the transverse loading of the cylindrical shell is to be zero, the governing equations for the homogenous material cylindrical shell with Reissner's effect are

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$$
\begin{cases}
D^{(k)}[\dfrac{\partial^2 j_x^{(k)}}{\partial x^2}+\dfrac{1-m}{2}\dfrac{\partial^2 j_x^{(k)}}{\partial y^2}+\dfrac{1+m}{2}\dfrac{\partial^2 j_y^{(k)}}{\partial x\partial y})+C^{(k)}(\dfrac{\partial w^{(k)}}{\partial x}-j_x^{(k)})+ \\
\dfrac{h^3}{24(1+m)}\dfrac{\partial E^{(k)}}{\partial y}(\dfrac{\partial j_x^{(k)}}{\partial y}+\dfrac{\partial j_y^{(k)}}{\partial x})=0 \\
D^{(k)}[\dfrac{\partial^2 j_y^{(k)}}{\partial y^2}+\dfrac{1-m}{2}\dfrac{\partial^2 j_y^{(k)}}{\partial x^2}+\dfrac{1+m}{2}\dfrac{\partial^2 j_x^{(k)}}{\partial x\partial y})+C^{(k)}(\dfrac{\partial w^{(k)}}{\partial y}-j_y^{(k)})+ \\
\dfrac{h^3}{12(1-m^2)}\dfrac{\partial E^{(k)}}{\partial y}(\dfrac{\partial j_y^{(k)}}{\partial y}+m\dfrac{\partial j_x^{(k)}}{\partial x})=0 \\
C^{(k)}(\nabla^2 w^{(k)}-\dfrac{\partial j_x^{(k)}}{\partial x}-\dfrac{\partial j_y^{(k)}}{\partial y})+k\dfrac{\partial^2 y^{(k)}}{\partial x^2}+\dfrac{5h}{12(1+m)}\dfrac{\partial E^{(k)}}{\partial y}(\dfrac{\partial w^{(k)}}{\partial y}-j_y^{(k)})=0
\end{cases}
\tag{2}
$$

$$
\begin{aligned}
&\nabla^2\nabla^2 y^{(k)}+kB^{(k)}\frac{\partial^2 w^{(k)}}{\partial x^2}-2\frac{\mathrm{d}E^{(k)}}{E^{(k)}\mathrm{d}y}(\frac{\partial^3 y^{(k)}}{\partial y^3}+\frac{\partial^3 y^{(k)}}{\partial x^2\partial y})+\frac{\mathrm{d}^2E^{(k)}}{E^{(k)}\mathrm{d}y^2}(m\frac{\partial^2 y^{(k)}}{\partial x^2}-\frac{\partial^2 y^{(k)}}{\partial y^2}) \\
&+2(\frac{\mathrm{d}E^{(k)}}{E^{(k)}\mathrm{d}y})^2(\frac{\partial^2 y^{(k)}}{\partial y^2}-m\frac{\partial^2 y^{(k)}}{\partial x^2})=0
\end{aligned}
\tag{3}
$$

where $D^{(k)}=\dfrac{E^{(k)}h^3}{12(1-m^2)}$, $C^{(k)}=\dfrac{5E^{(k)}h}{12(1+m)}$ and h is thickness of the cylindrical shell.

The crack tip stress field would be equipped with the same square root singularity as that of homogeneous materials when the material prosperities of different composite materials at the interfaces are continuous[4]. Therefore, the generalized displacements j_r, j_q, w can be expressed as follows

$$
\begin{aligned}
&j_x^{(k)}=\sum_{n=1}^{\infty} f_n^{(k)}(q)r^{\frac{n}{2}}, \quad j_y^{(k)}=\sum_{n=1}^{\infty} g_n^{(k)}(q)r^{\frac{n}{2}}, \\
&w^{(k)}=\sum_{n=1}^{\infty} j_n^{(k)}(q)r^{\frac{n}{2}}, \quad y^{(k)}=\sum_{n=1}^{\infty} y_n^{(k)}(q)r^{\frac{n+2}{2}}
\end{aligned}
\tag{4}
$$

where, $f_n^{(k)}(q)$ 、 $g_n^{(k)}(q)$ 、 $j_n^{(k)}(q)$ are eigen-functions and $k=\mathrm{I},\mathrm{II}$.

Substituting Eq.(4) into Eq.(2) and Eq.(3), the coefficients of $r^{-3/2}, r^{-1}, \ldots, r^{n/2-2}$ are linear independent, the each coefficient term must be zero. The obtained equations are solved and the eigen-function are derived as follows

$$
\begin{cases}
f_1^{(k)}=A_{12}^{(k)}\sin\dfrac{3q}{2}+A_{11}^{(k)}\cos\dfrac{3q}{2}+\dfrac{m+9}{m+1}A_{12}^{(k)}\sin\dfrac{q}{2}+\dfrac{3m-5}{m+1}A_{11}^{(k)}\cos\dfrac{q}{2} \\
g_1^{(k)}=A_{11}^{(k)}\sin\dfrac{3q}{2}-A_{12}^{(k)}\cos\dfrac{3q}{2}+\dfrac{m-7}{m+1}A_{11}^{(k)}\sin\dfrac{q}{2}+\dfrac{m-7}{m+1}A_{12}^{(k)}\cos\dfrac{q}{2} \\
j_1^{(k)}=A_{13}^{(k)}\sin\dfrac{q}{2} \\
y_1^{(k)}=A_{15}^{(k)}\sin\dfrac{3q}{2}+A_{14}^{(k)}\cos\dfrac{3q}{2}+A_{15}^{(k)}\sin\dfrac{q}{2}+3A_{14}^{(k)}\cos\dfrac{q}{2}
\end{cases}
\tag{5}
$$

$$
\begin{cases}
f_2^{(k)}=A_{21}^{(k)}\cos q+A_{22}^{(k)}\sin q \\
g_2^{(k)}=-A_{22}^{(k)}\cos q-mA_{21}^{(k)}\sin q \\
j_2^{(k)}=A_{23}^{(k)}\cos q \\
y_2^{(k)}=-A_{24}^{(k)}+A_{24}^{(k)}\cos 2q
\end{cases}
\tag{6}
$$

$$
\left\{
\begin{aligned}
f_3^{(k)} &= A_{31}^{(k)}\cos\frac{3q}{2} + A_{32}^{(k)}\sin\frac{3q}{2} + \frac{3(m+1)}{m-7}A_{31}^{(k)}\cos\frac{q}{2} + \frac{3(m+1)}{3m+11}A_{32}^{(k)}\sin\frac{q}{2} - \frac{4N_m A_{13}^{(k)}}{3m^2+8m-11}\cdot \\
&\quad \sin\frac{q}{2} - d_{2k}g\left[\frac{1}{4}A_{11}^{(k)}\sin\frac{5q}{2} - \frac{1}{4}A_{12}^{(k)}\cos\frac{5q}{2} - \frac{3m-13}{4(3m+11)}A_{11}^{(k)}\sin\frac{q}{2} + \frac{5m-43}{4(m-7)}A_{12}^{(k)}\cos\frac{q}{2}\right] \\
g_3^{(k)} &= \frac{7m-1}{3m+11}A_{32}^{(k)}\cos\frac{3q}{2} + \frac{3m-5}{m-7}A_{31}^{(k)}\sin\frac{3q}{2} - \frac{3(m+1)}{3m+11}A_{32}^{(k)}\cos\frac{q}{2} + \frac{3(m+1)}{m-7}A_{31}^{(k)}\sin\frac{q}{2} + \\
&\quad \frac{4N_m A_{13}^{(k)}}{8m+3m^2-11}\cos\frac{q}{2} + \frac{4(3m+1)N_m A_{13}^{(k)}}{3(8m+3m^2-11)}\cos\frac{3q}{2} + d_{2k}g\left[\frac{A_{12}^{(k)}}{4}\sin\frac{5q}{2} + \frac{3m-17}{3(m-7)}A_{12}^{(k)}\cdot\right. \\
&\quad \sin\frac{3q}{2} + \frac{3m-13}{4(m-7)}A_{12}^{(k)}\sin\frac{q}{2} + \frac{A_{11}^{(k)}}{4}\cos\frac{5q}{2} + \frac{9m+13}{3m+11}A_{11}^{(k)}\cos\frac{3q}{2} + \frac{21m+101}{4(3m+11)}\cdot \\
&\quad \left. A_{11}^{(k)}\cos\frac{q}{2}\right] \\
j_3^{(k)} &= \frac{2(m-1)A_{11}^{(k)}}{m+1}\cos\frac{q}{2} + \frac{2(m+7)A_{11}^{(k)}}{3(m+1)}\cos\frac{3q}{2} + \frac{2(m-1)A_{12}^{(k)}}{m+1}\sin\frac{q}{2} + A_{33}^{(k)}\sin\frac{3q}{2} - d_{2k}g\cdot \\
&\quad \frac{A_{13}^{(k)}}{12}\left(\cos\frac{3q}{2} + 3\cos\frac{q}{2}\right) - \frac{k}{C^{(1)}}\left(\frac{A_{15}^{(k)}}{16}\sin\frac{5q}{2} + \frac{A_{15}^{(k)}}{8}\sin\frac{q}{2} + \frac{15A_{14}^{(k)}}{16}\cos\frac{3q}{2} + \frac{15A_{14}^{(k)}}{8}\cos\frac{q}{2} + \right. \\
&\quad \left. \frac{3A_{14}^{(k)}}{16}\cos\frac{5q}{2}\right) \\
y_3^{(k)} &= A_{35}^{(k)}\sin\frac{5q}{2} + A_{34}^{(k)}\cos\frac{5q}{2} - A_{35}^{(k)}\sin\frac{q}{2} - 5A_{34}^{(k)}\cos\frac{q}{2} + \frac{kB^{(1)}}{32}A_{13}^{(k)}\left(\sin\frac{q}{2} + \right. \\
&\quad \left. \sin\frac{3q}{2}\right) + \frac{d_{2k}g}{4}\left(3A_{14}^{(k)}\sin\frac{3q}{2} + 3A_{14}^{(k)}\sin\frac{q}{2} - A_{15}^{(k)}\cos\frac{3q}{2} - 3A_{15}^{(k)}\cos\frac{q}{2}\right)
\end{aligned}
\right. \tag{7}
$$

$$
\left\{
\begin{aligned}
f_4^{(k)} &= A_{41}^{(k)}\cos 2q + A_{42}^{(k)}\sin 2q - \frac{m+1}{m-3}A_{41}^{(k)} + \frac{N_m A_{23}^{(k)}}{m^2+2m-3} \\
g_4^{(k)} &= A_{42}^{(k)}\cos 2q - \frac{2mA_{41}^{(k)}}{m+3}\sin 2q - \frac{m+1}{2}A_{42}^{(k)} - \frac{mA_{23}^{(k)}N_m}{m^2+2m-3}\sin 2q \\
j_4^{(k)} &= A_{43}^{(k)}\cos 2q - \frac{A_{22}^{(k)}}{2}\sin 2q - \frac{m-1}{4}A_{21}^{(k)} \\
y_4^{(k)} &= A_{44}^{(k)}\cos 3q + A_{45}^{(k)}\sin 3q - 3A_{45}^{(k)}\sin q - A_{44}^{(k)}\cos q
\end{aligned}
\right. \tag{8}
$$

where $d_{2k} = \begin{cases} 0, & k=1, \\ 1, & k=2, \end{cases}$ and $B_{ij}^{(k)}$ ($i = 1\,\mathbf{L}\,n;\ j = 1,2,3$)are undetermined coefficients.

Continuous condition

Substituting Eq.(5)-(8) into Eq.(4), the generalized displacement fields in homogenous material and FGMs region are obtained, and the stress fields will be obtained based on the relationship between the generalized displacement and stress. According to the following continuous conditions of stress

$$
s_y^{(1)} = s_y^{(2)},\ t_{xy}^{(1)} = t_{xy}^{(2)},\ t_{zy}^{(1)} = t_{zy}^{(2)} \qquad q=0 \tag{9}
$$

The generalized displacement fields in homogenous material and FGMs regions can be assembled finally.

Conclusions

The governing equations are derived for weak-discontinuous problems of FGMs cylindrical shells with Reissner’s effect. The higher order crack tip fields are obtained by assuming that the material parameter of FGM changes in linear function. All the singular items of the weak-discontinuity crack tip fields are continuous in the interface, but not in the higher order items. The solution provides the basis to solve problems of cylindrical shell fracture. Therefore, it possesses both important theoretical and engineering value.

Acknowledgements

The research is supported by the National Natural Science Foundation of China (No11172332.)

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