An analytical solution to the dynamic cracks in finite-width single-edge cracked strips

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Abstract. Using the method of complex analysis and through constructing appropriate conformal mapping, the paper has analyzed the plane elasticity problem of dynamic cracks in finite-width single-edged cracked strips, and provided an analytical solution to the crack-tip stress intensity factor. In addition, when the propagation velocity approaches zero, the dynamic solution can be restored to a static solution.

Introduction

In everyday life, materials with cracks are of common existence. In the theoretical field, there have already been studies regarding finite-height cracked strips of different materials. Article [1], for instance, provides an analytical solution to two semi-infinite collinear crack strips; article [2] provides an analytical solution to Type III cracks in piezoelectric ceramic strips; article [3] provides an analytical solution to asymmetrical fast propagating cracks in narrow bodies; article [4] provides an analytical solution to static cracks and fast propagating cracks in narrow bodies; article [5] provides a conformal mapping function and conformal maps the finite-width single-edged crack strip to the upper half plane, from which the stress intensity factors (SIFs) K_{I}, K_{II} of static cracks are obtained. In this paper we have extended the static problem of finite-width single-edged crack strips to a dynamic problem, through constructing a new conformal mapping function, and have provided the analytical solution to crack tip dynamic SIFs. Moreover, when the propagation velocity $V \to 0$, the dynamic solution can be restored to a static solution.

The dynamic crack problem in finite-width single-edge cracked tips

Let there be a fast propagating single-edge crack in a finite-with strip, under plane stress or plane strain state as shown in figure 1. In the fixed coordinate system (x_1, y, t), introduce Lame potentials $f(x_1, y, t)$ and y (x_1, y, t) [6], which gives

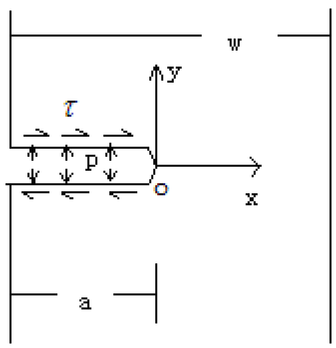

Fig. 1 Finite-width strip with dynamic crack

$$u_{x_1} = \frac{\partial f}{\partial x_1} + \frac{\partial y}{\partial y},\ u_y = \frac{\partial f}{\partial y} - \frac{\partial y}{\partial x_1} \tag{1}$$

Then the governing equation of plane elasticity dynamics is

$$\nabla^2 f = \frac{1}{C_1^2}\frac{\partial^2 f}{\partial t^2}, \ \nabla^2 y = \frac{1}{C_2^2}\frac{\partial^2 y}{\partial t^2} \tag{2}$$

in which ∇^2 is the two dimensional Laplace operator and C_1 and C_2 represent wave velocities of the longitudinal wave and the transverse wave, this gives

$$C_1 = \left(\frac{l+2m}{r}\right)^{1/2}, \ C_2 = \left(\frac{m}{r}\right)^{1/2} \tag{3}$$

l, m and r are Lame coefficients and the mass density of the material.

Let the crack with a speed of constant V propagate along the ox_1 direction, as shown in figure 1. Use the Galileo transformation

$$x = x_1 - Vt, \ y = y \tag{4}$$

The wave equation set is

$$\nabla_1^2 f(x, y_1) = 0, \ \nabla_2^2 y(x, y_2) = 0 \tag{5}$$

in which

$$\begin{cases} \nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y_1^2}, \qquad \nabla_2^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y_2^2} \\ y_1 = a_1 y, \qquad y_2 = a_2 y \\ a_1 = \left(1 - V^2/C_1^2\right)^{1/2}, \quad a_2 = \left(1 - V^2/C_2^2\right)^{1/2} \end{cases} \tag{6}$$

The solution of equation set (5) can be given in the plural as follows

$$f(x, y_1) = F_1(z_1) + \overline{F_1(z_1)}, \ y(x, y_2) = \mathrm{i}\left[F_2(z_2) + \overline{F_2(z_2)}\right] \tag{7}$$

Here $F_1(z_1)$ and $F_2(z_2)$ are, respectively, analytic functions of complex variables $z_1 = x + \mathrm{i}y_1$, $z_2 = x + \mathrm{i}y_2$, $\overline{F_1(z_1)}$ and $\overline{F_2(z_2)}$ represent its complex conjugate functions.

Bring in signs

$$\Phi(z_1) = \frac{\mathrm{d}F_1(z_1)}{\mathrm{d}z_1} = F_1'(z_1), \ \Psi(z_2) = \frac{\mathrm{d}F_2(z_2)}{\mathrm{d}z_2} = F_2'(z_2) \tag{8}$$

Then stress and displacement can be expressed as

$$\begin{cases} s_{xx} + s_{yy} = 4m\left(a_1^2 - a_2^2\right)\mathrm{Re}\,\Phi'(z_1) \\ s_{yy} - s_{xx} + 2\mathrm{i}s_{xy} = 2m\Big[(1-a_1)^2\Phi'(z_1) + (1+a_1)^2\overline{\Phi'(z_1)} + \\ (1-a_2)^2\Psi'(z_2) - (1+a_2)^2\overline{\Psi'(z_2)}\Big] \end{cases} \tag{9}$$

$$u_x + u_y = (1-a_1)\Phi(z_1) + (1+a_1)\overline{\Phi(z_1)} + (1-a_2)\Psi(z_2) - (1+a_2)\overline{\Psi(z_2)} \tag{10}$$

When crack surface is under uniform internal pressure (Type I problem), we have the following boundary conditions

$$\begin{cases} s_{yy} = s_{xy} = 0, & y = \pm\infty, \quad -a < x < w-a \\ s_{xx} = s_{xy} = 0, & x = -a, \quad x = w-a, \quad -\infty < y < \infty \\ s_{yy} = f_1(x), \quad s_{xy} = 0, & y = \pm 0, \quad -a < x < 0 \end{cases} \tag{11}$$

in which $f_1(x)$ is a known arbitrary function, here $f_1(x) = -p$ = constant.

Therefore it comes down to the following basic question:

Suppose the crack moves along the x-axis in constant velocity V, when the corresponding boundary condition (11) is satisfied on the crack and the finite-width boundary (Type I problem), solve for the analytic function $\Phi(z_1)$ on the z_1 plane and the analytic function $\Psi(z_2)$ on the z_2 plane.

In order to find the solution for equation (5) under boundary condition (11), we construct the following conformal mapping function

$$z_1/a_1,\ z_2/a_2 = w(z) = \frac{2w}{p} arc\tan\left[\sqrt{\tan^2\frac{pa}{2w} - z^2}\right] - a \tag{12}$$

And fix the analysis branch for the $\ln 1 = 0$, this transformation simultaneously maps the areas on the z_1 plane and the z_2 plane, which become the upper-half plane of the z plane, and the crack is mapped to become a part of the x axis.

On the z plane the corresponding problem of boundary values comes down to solving the following functional equation set[6]:

$$\begin{cases} \dfrac{1}{2pi}\displaystyle\int_\Delta G_1(s)\frac{ds}{s-z} + \frac{1}{2pi}\int_\Delta \frac{w(s)}{\overline{w'(s)}}\overline{G_1(s)}\frac{ds}{s-z} = \frac{1}{2m}\frac{1}{2pi}\int_\Delta w'(s)p\frac{ds}{s-z} \\ \dfrac{1}{2pi}\displaystyle\int_\Delta G_2(s)\frac{ds}{s-z} - \frac{1}{2pi}\int_\Delta \frac{w(s)}{\overline{w'(s)}}\overline{G_2(s)}\frac{ds}{s-z} = 0 \end{cases} \tag{13}$$

in which

$$\begin{cases} G_1(z) = a_1\left(1+a_2^2\right)\Phi_*'(z) - 2a_2^2\Psi_*'(z) \\ G_2(z) = 2a_1^2\Phi_*'(z) - a_2\left(1+a_2^2\right)\Psi_*'(z) \end{cases} \tag{14}$$

and

$$\begin{cases} \Phi(z_1) = \Phi\left[w(z)\right] = \Phi_*(z), & \Psi(z_2) = \Psi\left[w(z)\right] = \Psi_*(z) \\ \Phi'(z_1) = \Phi_*'(z)/a_1 w'(z), & \Psi'(z_2) = \Psi_*'(z)/a_2 w'(z) \end{cases} \tag{15}$$

Below we solve for $G_1(z)$

From the first equation of equation (13) we obtain

$$G_1(z) = \frac{1}{2m}\frac{p}{2pi}\int_{-h}^{h} w'(s)\frac{ds}{s-z} \tag{16}$$

in which

$$h = \sqrt{\tan^2\frac{pa}{2w} - \tan^2\frac{pa(a_1-1)}{2wa_1}} \tag{17}$$

By calculation we can obtain

$$G_1(0)=\frac{pw}{2mp^2}\cos\frac{pa}{2w}\log\left[\frac{1}{2}\csc^2\frac{pa}{2w}\left(1+\cos\frac{pa}{w}-2\cos\frac{pa(a_1-1)}{wa_1}-\right.\right.$$

$$\left.\left.4\cos^2\frac{pa(a_1-1)}{2wa_1}\sqrt{\tan^2\frac{pa(a_1-1)}{2wa_1}\cos^2\frac{pa}{2w}}\sqrt{\tan^2\frac{pa(a_1-1)}{2wa_1}-\tan^2\frac{pa}{2w}}\right)\right] \tag{18}$$

$$G_2(z)=0 \tag{19}$$

in which

$$\begin{cases} z=w^{-1}(z_1/a_1)=w^{-1}(z_2/a_2) \\ h=z\,|_{z_1^-}=z\,|_{z_2^-}, \qquad -h=z\,|_{z_1^+}=z\,|_{z_2^+} \\ z_1^-=(-a,0^-), \qquad z_2^-=(-a,0^-), \qquad z_1^+=(-a,0^+), \quad z_2^+=(-a,0^+) \end{cases} \tag{20}$$

Make simultaneous the equations (18), (19) and (14), we get

$$\begin{cases} \Phi_*'(z)=\dfrac{-(1+a_2^2)/a_1}{4a_1a_2-(1+a_2^2)^2}G_1(z)=\dfrac{-(1+a_2^2)/a_1}{4a_1a_2-(1+a_2^2)^2}G_1[w'(z_1)] \\ \Psi_*'(z)=\dfrac{-2a_1/a_2}{4a_1a_2-(1+a_2^2)^2}G_1(z)=\dfrac{-2a_1/a_2}{4a_1a_2-(1+a_2^2)^2}G_1[w'(z_2)] \end{cases} \tag{21}$$

Below we calculate the important physical quantity - the dynamic stress intensity factor [6]:
The crack-tip dynamic asymptotic stress field is

$$s_{xx}+s_{yy}=\frac{-2(a_1^2-a_2^2)}{\sqrt{2pr_1}}\cdot f\cdot K_{\mathrm{I}}\cos\frac{q_1}{2} \text{ (Type I problem)} \tag{22}$$

$$s_{xx}+s_{yy}=\frac{2(a_1^2-a_2^2)}{\sqrt{2pr_1}}\cdot g\cdot K_{\mathrm{II}}\sin\frac{q_1}{2} \text{ (Type II problem)} \tag{23}$$

in which $z_1=r_1e^{iq_1}$, $r_1<<a$ and

$$f=(1+a_2^2)\Big/\left[4a_1a_2-(1+a_2^2)^2\right],\ g=2a_2\Big/\left[4a_1a_2-(1+a_2^2)^2\right] \tag{24}$$

Here

$$K_{\mathrm{I}}=\lim_{r_1\to 0}\sqrt{2pr_1}\,s_{yy}(r_1,0,t),\ K_{\mathrm{II}}=\lim_{r_1\to 0}\sqrt{2pr_1}\,s_{xy}(r_1,0,t) \tag{25}$$

Bring in the combined stress intensity factor

$$K\equiv fK_{\mathrm{I}}-\mathrm{i}gK_{\mathrm{II}} \tag{26}$$

Then equations (22), (23) can be written as

$$s_{xx}+s_{yy}=2(a_1^2-a_2^2)\mathrm{Re}\left\{K/\sqrt{2pz_1}\right\} \tag{27}$$

Comparing the equation to equation (9), derive

$$K=2m\lim_{z_1\to 0}\sqrt{2pz_1}\,\Phi_*'(z_1)=4m\sqrt{p}\lim_{V\to 0}\Phi_*'(z)\Big/\sqrt{w''(z)} \tag{28}$$

From equations (21) and (24) we know that

$$\lim_{V \to 0} \Phi_*'(z) = \frac{-f}{a_1} G_1(0) \tag{29}$$

Therefore from equations (12), (18) and (28) we can obtain

$$K = \frac{-4m\sqrt{p}}{a_1} \cdot f \cdot G_1(0) \Big/ \sqrt{-\frac{2w}{p} c\tan\frac{pa}{2w}\cos^2\frac{pa}{2w}} \tag{30}$$

Then

$$K_{\text{I}} = \frac{-4m\sqrt{p}}{a_1} G_1(0) \Big/ \sqrt{-\frac{2w}{p} c\tan\frac{pa}{2w}\cos^2\frac{pa}{2w}} \tag{31}$$

Similarly, when the crack surface is under uniform shear and shearing strength is $-t$, we can obtain

$$K_{\text{II}} = \frac{-4a_2 m\sqrt{p}}{a_1^2} G_2(0) \Big/ \sqrt{-\frac{2w}{p} c\tan\frac{pa}{2w}\cos^2\frac{pa}{2w}} \tag{32}$$

in which

$$\begin{aligned} G_2(0) = {} & \frac{tw}{2mp^2}\cos\frac{pa}{2w}\log\left[\frac{1}{2}\csc^2\frac{pa}{2w}\left(1+\cos\frac{pa}{w} - 2\cos\frac{pa(a_1-1)}{wa_1} - \right.\right. \\ & \left.\left. 4\cos^2\frac{pa(a_1-1)}{2wa_1}\sqrt{\tan^2\frac{pa(a_1-1)}{2wa_1}\cos^2\frac{pa}{2w}}\sqrt{\tan^2\frac{pa(a_1-1)}{2wa_1} - \tan^2\frac{pa}{2w}}\right)\right] \end{aligned} \tag{33}$$

When $V \to 0$, equation (31) and (32) are restored, respectively, to static crack stress intensity factors $K_{\text{I}} = \sqrt{2}p\sqrt{w\tan\frac{pa}{2w}}$, $K_{\text{II}} = \sqrt{2}t\sqrt{w\tan\frac{pa}{2w}}$, which is identical to the results obtained in article [5].

Conclusions

Regarding the dynamic crack problem in finite-width single-edge cracked strips, the conformal mapping equation (12) provided by this article is a transcendental function, using conformal mapping to simplify the complicated crack problem in order to obtain a solution, and the calculation method is relatively simple. The method in this article is an extension to the Muskhelishvili[7] complex potential method, expanding the scope of application and enriching the content of the latter. By constructing new conformal mapping functions, this study has analyzed the stress intensity factor of dynamic cracks in finite-width single-edged cracked strips, and provided an analytical solution to the crack-tip stress intensity factor; furthermore, when the propagation velocity approaches zero, the dynamic solution can be restored to a static solution. The research in this article is theoretically significant and practically applicable in solving many actual problems of engineering fracture.

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