On Generalized Order of Vector Dirichlet Series of Fast Growth

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Abstract—The concept of vector valued Dirichlet series was introduced by B. L. Srivastava [2] who characterized the growth of entire functions represented by these series. In this paper we introduce the generalized order of analysis functions fast growth.

Keywords—Vector valued dirichlet series; Analysis functions; Generalized order; Fast growth.

I. INTRODUCTION

Let

\[ f(s) = \sum_{n=0}^{+\infty} a_n e^{-\lambda_n s} \quad (s = \sigma + it, \sigma, t \in \mathbb{R}) \tag{1} \]

where \(a_n\)'s belong to a complex commutative Banach algebra \(B\) with identity element \(1 = \omega\) and \(\lambda_n\)'s satisfy the conditions

\[ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \uparrow +\infty \]

Then the vector valued Dirichlet series in (1) represents an analytic function \(f(s)\) in right plane (see [1]). For the vector valued analytic function \(f(s)\) defined as above by (1) the maximum modulus, the maximum term and the index of maximum term are defined as

\[ M(\sigma, f) = \sup_{-\infty < t < +\infty} \|f(\sigma + it)\| \]

\[ m(\sigma, f) = \max_{n \in \mathbb{N}} \|a_n\| e^{-\lambda_n \sigma} \]

The order \(\rho\) of \(f(s)\) is defined as

\[ \rho = \lim_{\sigma \to 0} \frac{\log^+ M(\sigma, f)}{-\log \sigma} \]

We shall call the vector valued analytic function \(f(s)\) to be of fast growth if the order \(\rho = \infty\). We obtain the characterization of growth parameters in the context of generalized order of vector valued Dirichlet series of fast growth.

Let \(\Delta_0\) be the class of all functions \(\beta\) satisfying the following two conditions:

(i) \(\beta(x)\) is defined on \([a, \infty), a > 0\), and is positive, strictly increasing, differentiable and tends to \(\infty\) as \(x \to \infty\);

(ii) \(\frac{d\beta(x)}{d \log x} = o(1) \) as \(x \to \infty\).

For a vector valued analytic function \(f(s)\) given by (1) and \(\beta(x) \in \Delta_0\), set

\[ \rho(\beta, f) = \lim_{\sigma \to 0} \frac{\beta(\log^+ m(\sigma, f))}{-\log \sigma} \]

Then \(\rho(\beta, f)\) will be called, respectively, \(\beta\)-order of \(f(s)\). To avoid some trivial cases we shall assume throughout that \(M(\sigma, f) \to \infty\) as \(\sigma \to 0\).

II. MAIN RESULTS

Lemma 2.1 If the vector valued Dirichlet series given by (1) satisfies (2), then

\[ \frac{1}{(1 + \varepsilon)} M(\sigma, f) \leq K(\varepsilon) m(\sigma(1 - \varepsilon), f) \frac{1}{\sigma} \]

where \(K(\varepsilon)\) is a positive number of \(\varepsilon\) and \(\beta(\sigma)\).

Proof: From the second equation of (2), for given \(\varepsilon > 0\), there exists an \(n_0 = n_0(\varepsilon)\), such that for \(n > n_0\), \(n < (D + \varepsilon)\lambda_0\). Taken \(\sigma < 0\), \(\varepsilon \in (0, -\sigma_0)\), we have

\[ M(\sigma, f) \leq \sum_{n=n_0}^{n_0} \|a_n\| e^{-\lambda_n \sigma} + \sum_{n=n_0+1}^{\infty} \|a_n\| e^{-\lambda_n (1 - \varepsilon) \sigma} e^{-\lambda_n \sigma} \]

\[ < n_0 m(\sigma, f) + m((1 - \varepsilon)\sigma, f) \frac{1}{1 - e^{-\sigma/(D + \varepsilon)}} \]

The lemma now follows from above and the well known relation \(m(\sigma, f) \leq M(\sigma, f)\).

Theorem 2.1 If the vector valued Dirichlet series given by (1) satisfies (2), then

\[ \rho(\beta, f) = \lim_{\sigma \to 0} \frac{\beta(\log^+ m(\sigma, f))}{-\log \sigma} \]

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Proof: By the first inequality of Lemma 2.1, this gives, since $\beta \in \Delta_0$, that

$$
\lim_{\sigma \to 0} \frac{\beta(\log^+ m(\sigma, f))}{-\log \sigma} \leq \lim_{\sigma \to 0} \frac{\beta(\log^+ M(\sigma, f))}{-\log \sigma} \quad (3)
$$

By the second inequality of Lemma 2.1, we have

$$
\log^+ M_\sigma(\sigma, F) \leq \log^+ m(\sigma(1 - \epsilon), f) - \log \sigma + O(1)
$$

$$
\leq C \log^+ m(\sigma(1 - \epsilon), f) \cdot (-\log \sigma) \quad (4)
$$

For all $\sigma(\sigma > 0)$ sufficiently close to 0. Here $C$ is a constant. Now, (4) gives

$$
\beta(\log^+ M(\sigma, f))
$$

$$\leq \beta(\log^+ m(\sigma(1 - \epsilon), m)) + \log((-\log \sigma)^C) \cdot \frac{d\beta(x)}{d\log x} \bigg|_{x = x'(\sigma)}
$$

Where $\log^+ \mu(\sigma(1 - \epsilon), F) < x'(\sigma) < C \log^+ \mu(\sigma(1 - \epsilon), F)^+$.

$(-\log \sigma)$. This easily gives

$$
\lim_{\sigma \to 0} \frac{\beta(\log^+ M(\sigma, f))}{-\log \sigma} \leq \lim_{\sigma \to 0} \frac{\beta(\log^+ m(\sigma, f))}{-\log \sigma} \quad (5)
$$

The theorem follows from (3) and (5).

Theorem 2.2 If the vector valued Dirichlet series given by (1) satisfies (2) and has $\beta$-order $\rho(\beta, f)$, then

$$
\rho(\beta, f) = \lim_{n \to \infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log^+ \log \| a_n \|}
$$

Proof: Let $\lim_{n \to \infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log^+ \log \| a_n \|} = \theta$. Clearly $0 \leq \theta \leq \infty$. First let $0 < \theta < \infty$. Then, for $0 < \epsilon < \theta$ there exist a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$
\log \| a_{n_k} \| > \lambda_{n_k} \exp\{-\frac{\beta(\lambda_{n_k})}{\theta - \epsilon}\}, \quad k = 1, 2, 3, \ldots.
$$

By Lemma 2.1, we have

$$
\log^+ M(\sigma, f) \geq \log^+ m(\sigma, f) \geq \log \| a_{n_k} \| - \sigma \lambda_{n_k}
$$

$$
> \lambda_{n_k} \exp\{-\frac{\beta(\lambda_{n_k})}{\theta - \epsilon}\} - \sigma \lambda_{n_k} \quad (6)
$$

For $k = 1, 2, 3, \ldots$, set $\sigma_k = \frac{1}{2} \exp\{-\frac{\beta(\lambda_{n_k})}{\theta - \epsilon}\}$. Putting, in particular, $\sigma = \sigma_k$ in (6), we get

$$
\log^+ M(\sigma_k, f) > \frac{1}{2} \lambda_{n_k} \exp\{-\frac{\beta(\lambda_{n_k})}{\theta - \epsilon}\} = \lambda_{n_k} \sigma_k,
$$

$$\log(1) \log^+ M(\sigma, f) > \beta(\lambda_{n_k}) = (\theta - \epsilon) \log(\frac{1}{2\sigma_k})
$$

Since $\beta \in \Delta_0$, we have

$$
\beta(\frac{1}{\sigma_k} \log^+ M(\sigma_k, f))
$$

$$= o(1) \log(\frac{1}{\sigma_k}) \log^+ M(\sigma_k, f)
$$

$$= o(1) \log(\frac{1}{\sigma_k}) + o(1) \log^+ M(\sigma_k, f)
$$

$$= \log^+ (\frac{1}{\sigma_k}) \frac{d\beta(x)}{d\log x} \bigg|_{x = x'(\sigma_k)} + \beta(\log^+ M(\sigma_k, f)) \quad (7)
$$

where $\log^+ M(\sigma_k, f) < x'(\sigma_k) < \frac{1}{\sigma_k} \log^+ M(\sigma_k, f)$.

By (7), we have

$$
\beta(\log^+ M(\sigma_k, f)) + \log(\frac{1}{\sigma_k}) \frac{d\beta(x)}{d\log x} \bigg|_{x = x'(\sigma_k)} + \beta(\log^+ M(\sigma_k, f)) > (\theta - \epsilon) \log(\frac{1}{2\sigma_k})
$$

Since $\beta \in \Delta_0$, dividing by $\log(\frac{1}{\sigma_k})$ and passing to limits, we get

$$
\rho(\beta, f) \geq \theta \quad (8)
$$

(8) is obvious for $\theta = 0$. For $\theta = \infty$, the above arguments with an arbitrarily large number in place of $\theta - \epsilon$ give $\rho(\beta, F) = \infty$.

To prove the reverse inequality, since there is nothing to prove if $\theta = \infty$, we may assume that $\theta < \infty$. Then, given $\epsilon > 0$ and for all $n > n_k = n_k(\epsilon)$ we have
Now the second equation of (2) holds, we have $n < D\lambda_n$ for all $n > n_2 = n_2(D)$, where $D > D$ is a fixed constant. Let $n_3 = \max\{n_1, n_2\}$, then from (9), we have

$$M(\sigma, f) \leq \sum_{n=n_3+1}^{\infty} \exp\{-\beta(\lambda_n)\} - \lambda_n \sigma$$

For every $\sigma(\sigma > 0)$ we define $n(\sigma)$ as

$$\lambda_n(\sigma) \leq \beta^{-1}(-(\theta + \epsilon) \log(\sigma/2)) < \lambda_n(\sigma+1)$$

For $\sigma(\sigma > 0)$ is sufficiently close to 0, we have $n(\sigma) > n_3$. Thus, we have

$$\sum_{n=n_3+1}^{\infty} \exp\{-\beta(\lambda_n)\} - \lambda_n \sigma < \sum_{n=n_3+1}^{\infty} \exp\{-\sigma(\sigma+1)\} \cdot \frac{\sigma(\sigma+1)}{2D} \leq H(n(\sigma)).$$

Now, the equation $H(n(\sigma)) = -\frac{\sigma(\sigma+1)}{2D} \log(\sigma/2) + \frac{2}{\sigma} + o(1)$

$$< -\frac{1}{x(\sigma)} \beta^{-1}((\theta + \epsilon) \log(x(\sigma))) + \log(x(\sigma)) + o(1),$$

where $x(\sigma) = \frac{2}{\sigma}$. Clearly $x(\sigma) \to \infty$ as $\sigma \to 0$.

Since $\beta \in \Delta_0$, it follows that $\beta^{-1}(\theta \log(x(\sigma))) > (x(\sigma))^2$ for all $\sigma$ sufficiently close to 0. This shows that

$$\log H(n(\sigma)) \to \infty \text{ as } \sigma \to 0 \text{ or }$$

$$H(n(\sigma)) \to 0 \text{ as } \sigma \to 0.$$  \hspace{1cm} (10)

Consider the function

$$\phi(x) = x \exp\{-\frac{\beta(x)}{\theta + \epsilon}\} - x \sigma$$

Taking derivative of $\phi(x)$ and setting it equal to 0 we get

$$\phi'(x) = \exp\{-\frac{\beta(x)}{\theta + \epsilon}\} \frac{1}{\theta + \epsilon} - \frac{1}{\theta + \epsilon} \log x = 0$$

$$\frac{d\beta(x)}{d\log x} = o(1)$$

Since $\frac{1}{\theta + \epsilon} - \frac{1}{\theta + \epsilon} \log x < 2$ for $x > x_0$.

Let $\lambda_n$ be a fixed $\lambda_n$ greater than $x_0$ and $\lambda$, then $\phi'(\lambda_n) > 0$ for $0 < \sigma < \sigma_1$. Also $\phi'(\lambda_{n(\sigma)+1}) < 0$ for all $0 < \sigma < \sigma_2$. Now for $0 < \sigma < \sigma_0 = \min(\sigma_1, \sigma_2)$, we denote by $x_0(\sigma)$ the point where

$$\phi(x_0(\sigma)) = \max_{\lambda_n \leq \lambda \leq \lambda_{n(\sigma)+1}} \phi(x).$$

Then $\lambda_n < x_0(\sigma) < \lambda_{n(\sigma)+1}$ and

$$x_0(\sigma) = \beta^{-1}(-(\theta + \epsilon) \log(\sigma/2))$$

$$d(\sigma) = \frac{1}{\theta + \epsilon} \frac{d\beta(x)}{d\log x} \bigg|_{x=x_0(\sigma)}$$

Where $\max_{n_0 \leq \sigma \leq \sigma_{n_0+1}} \{ \| a_n \| e^{-\lambda \sigma} \} \leq \exp\{\phi(x_0(\sigma))\}$

$$\leq \exp\{\beta^{-1}(-(\theta + \epsilon) \log(\sigma/2)) \cdot \frac{\sigma}{1-d(\sigma)} \cdot \frac{\sigma}{1-d(\sigma)} \cdot \frac{\sigma}{1-d(\sigma)} \cdot \frac{\sigma}{1-d(\sigma)} \}.$$
\[ \leq P(n_0) + \bar{D} \beta^{-1} \left( - (\theta + \varepsilon) \log \frac{\sigma}{2} \right). \]
\[ \exp \{ \sigma \beta^{-1} \left( - (\theta + \varepsilon) \log \frac{\sigma}{2} \right) \} + o(1) \]

Or
\[ \log' M_u (\sigma, F) \leq \sigma \beta^{-1} \left( - (\theta + \varepsilon) \log \frac{\sigma}{2} \right) (1 + o(1)) . \]

Since \( \beta \in \Delta_0 \), this easily gives
\[ \rho(\beta, F) \leq \theta \quad (12) \]

The theorem now follows from (8) and (12).

REFERENCES


