A Low Order Nonconforming Mixed Finite Element Scheme for Nonlinear Integro-differential Equations of Pseudo-hyperbolic Type

Xianzhi Li, Kaiguang Zhang, Hongling Meng
Mathematics College, Zhengzhou normal University, Zhengzhou, china

Abstract—In this paper, a low order triangular nonconforming mixed finite element $(\Theta + \Theta + \Theta')$ scheme was studied for the nonlinear integro-differential equations of pseudo-hyperbolic type. By utilizing the properties of the interpolation, mean-value and derivative delivery techniques, the corresponding convergence analysis, the optimal error estimates of the original variable in $L^2$-norm and intermediate variable $P$ in $L^2$-norm are obtained.

Keywords—nonlinear pseudo-hyperbolic integro-differential equation; triangular nonconforming finite element; new mixed finite element scheme; optimal error estimate.

I. INTRODUCTION

In recent years, along with the widely application in viscoelastic mechanics, nuclear reaction kinetics and biomechanics, the research of the integro-differential equations with pseudo-hyperbolic type attracts much attention. We consider the following integro-differential equations with pseudo-hyperbolic type on a convex bounded region $\Omega$ with continuous boundary $\partial\Omega$,

$$
\begin{align*}
\mathbf{u}_t - \nabla \cdot (\mathbf{b}(\mathbf{x},t) \nabla \mathbf{u}) + \mathbf{f}(\mathbf{x},t) & \in \mathbf{B}, & \mathbf{u}(\mathbf{X};t) + \mathbf{J} \mathbf{I} \mathbf{u} = 0, & \mathbf{u}(\mathbf{X};T) = \mathbf{u}_0, & \mathbf{u}(\mathbf{X};0) = \mathbf{u}_0, \\
\mathbf{u}(\mathbf{X};t) & \in \mathbf{H}, & \mathbf{B} & \subset \mathbf{H}, & \mathbf{I} & \in \mathbf{H}.
\end{align*}
$$

Where $\Omega \subset \mathbb{R}^d$, $J \subset (0,T)$, for Arbitrarily $T$, $T \subset (0,\infty)$.

The functions $\mathbf{b}(\mathbf{x},t)$ and $\mathbf{f}(\mathbf{x},t)$ respectively satisfy

$$
0 < b_i < b_{i+1} < \infty, \quad 0 < c_i < c_{i+1} < \infty,
$$

where $b_0$, $b_1$, $c_0$ and $c_1$ are constants, $b$, $c$, $f$, $u_0$ and $u_1$ are known smooth functions.

II. MIXED FINITE ELEMENT FORMULATION

Assume $\Gamma_1$ is a regular triangular subdivision to domain $\Omega$, for arbitrary $\mathbf{K} \in \Gamma_1$, the coordinates of three vertices are $u_i(x,y), i = 1,2,3$ respectively, the three edges are $l_i = \overline{u_i u_{i+1}}$, $l_i = \overline{u_i u_{i+2}}$, $l_i = \overline{u_i u_{i+3}}$, respectively, where $h_k$ is the longest diameter of the unit $\mathbf{K}$. Define the finite element spaces $V^\mathbf{v}$ and $M^\mathbf{w}$ respectively are

$$
V^\mathbf{v} = \{v' \in L^2(\Omega) \mid \int_\Omega v' \, d\mathbf{x} = 0, \forall \mathbf{K} \in \Gamma_1\},
$$

and

$$
M^\mathbf{w} = \{w' \in L^2(\Omega) \mid \int_\Omega w' \, d\mathbf{x} = 0, \forall \mathbf{K} \in \Gamma_1\},
$$

where $P(K) = \text{span}\{1,x,y\}$, $P(K) = \text{span}\{1\} \times \text{span}\{1\}$, $[v']$ is a jump value of edge $F$.

Define the interpolating operators $I^\mathbf{v}$ and $I^\mathbf{w}$ respectively are

$$
I^\mathbf{v} : u \in L^2(\Omega) \to I^\mathbf{v} u \in V^\mathbf{v}, \quad \int_\Omega \mathbf{v} \cdot \mathbf{n} \, d\mathbf{x} = 0, i = 1,2,3,
$$

and

$$
I^\mathbf{w} : p \in L^2(\Omega) \to I^\mathbf{w} p \in M^\mathbf{w}, \quad \int_\Omega (p - I^\mathbf{w} p) \, d\mathbf{x} = 0.
$$

The intermediate variables to decrease the smooth degree of finite element space in order to solve these problems, but the two approximation space used by these methods need satisfy $B - B$ needs satisfaction needs satisfaction condition. For second order elliptic problem, elliptic problem [9,10] propose a new scheme which means when the two finite element spaces satisfy a simple inclusion relation, $B - B$ needs satisfaction needs satisfaction condition is certainly true. [11,12] using the high precision technique, for second order elliptic problem and linear elastic problem, study the overconvergence property on $\Omega$ conforming finite element space, these methods is applied to Sobolev equation in [13]. Using nonconforming linear triangle element Crouzeix-Raviart, [14] applies these methods to hyperbolic type integro-differential equation to obtain optimal error estimate. Using nonconforming $L^2$-element, [15] applies these methods to parabolic type equation to obtain convergence property and extrapolation. Using triangle conforming element, [16] applies these methods to parabolic type integro-differential equation to obtain overapproximation and overconvergence property.

This paper applies the scheme in [17] to formulation(1), by using the properties of the interpolation, mean-value and derivative delivery techniques, to analyze the convergence property, the optimal error estimates of the original variable $u$ in $H^r$-norm and intermediate variable $P$ in $E$-norm are obtained.
It is obviously $P_{x}F_{z}$ is the module on $V^*$, and $P_{x}P_{z} = \sum_{i=1}^{n} a_{i}^{2}$, so we have $P_{x}P_{z} = \sum_{i=1}^{n} a_{i}^{2}$, for arbitrary $\phi_{x}^{*} \in M$, we have

$$\sum_{i=1}^{n} [\gamma(x_{i})]^{2} = \sum_{i=1}^{n} [\gamma(x_{i})]^{2}$$

(2)

According to [19,20], for arbitrary $P_{x} \in (H^{2}(\Omega))^{2}$ and $\phi_{x}^{*} \in V^{*}$, the following inequality is true

$$\sum_{i=1}^{n} [\gamma(x_{i})]^{2} \leq c \sum_{i=1}^{n} [\gamma(x_{i})]^{2}$$

where $c$ is a positive constant, and independent of $h$. According to [2], the following inequality is true

$$\sum_{i=1}^{n} [\gamma(x_{i})]^{2} \leq c \sum_{i=1}^{n} [\gamma(x_{i})]^{2}$$

(4)

where $\phi$ is an integrable function on $[0,1]$, for arbitrary $\phi(x)$.

For constructing the mixed finite element scheme of problem (1), we introduce the adjoint vectors function of $\pi$, $P = -\nabla u - \nabla \int_{\Omega}(u(x),y)\nabla u(x,y)dx$, and then the problem could re-write as the following first order system,

$$\begin{align*}
(u_{x}, v) - (P, v) = & \langle (f, u_{x}), v \rangle, \\
(P, q) + (\nabla u, q) + (\nabla u_{x}, q) &= 0, \quad \forall q \in M, \exists J, \\
(u, 0) = & u_{0}(x), \quad \forall x \in \Omega.
\end{align*}$$

(5)

Its well known formula is extracting $[u_{0}(x), f_{0}(x)] : [0, 1] \rightarrow V \times M$ to satisfy

$$\begin{align*}
(u_{x}, v) - (P, v) &= \langle (f, u_{x}), v \rangle, \\
(P, q) + (\nabla u, q) + (\nabla u_{x}, q) &= 0, \quad \forall q \in M, \exists J, \\
(u, 0) = & u_{0}(x), \quad \forall x \in \Omega.
\end{align*}$$

(6)

Taking note of $\nabla v$ and $\phi_{z}$ are constants in every unit, $(\nabla v, \phi_{z}) = 0$, and hence, the error equation could re-write as

$$\begin{align*}
(\xi_{x}, v) - (\phi_{x}, v) &= (\xi_{x}, v) - (\phi_{x}, v) - \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(\theta, v) + (\nabla v, q) + (\nabla v_{x}, q) &= (\theta, v) + (\nabla v, q) + (\nabla v_{x}, q) - (\nabla \xi_{x}, q), \quad \forall \phi_{z} \in M
\end{align*}$$

(12)

Let $\phi = \nabla v$, the second equation in (12) becomes

$$\begin{align*}
(\nabla v, q) + (\nabla v_{x}, q) &= - (\nabla \xi_{x}, q), \\
(\nabla v_{x}, q) &= - (\nabla \xi_{x}, q) - \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(13)

For estimating the term in the right side, we introduce $\nabla v_{x}$ as $\nabla v_{x} = \frac{1}{|K|} \int_{K} \nabla v_{x} dV$, $[\gamma(x_{i})]^{2}$ is defined, which mean the unit $K$ as $\nabla v_{x} = \frac{1}{|K|} \int_{K} \nabla v_{x} dV$, and then we have $|\phi_{z} - \phi_{x}| \leq c \beta |P_{x}P_{z}|$, using the mean technique and Young inequality, be aware of $\nabla(\xi_{x}, q) = \sum_{i=1}^{n} [\gamma(x_{i})]^{2}$, then we have

$$\begin{align*}
\nabla(\xi_{x}, q) &= \sum_{i=1}^{n} (b - \nabla \xi_{x}, q)_{y_{i}} \leq c \beta |P_{x}P_{z}|
\end{align*}$$

(14)

where $V = H_{0}^{1}(\Omega), M = (L^{2}(\Omega))^{2}$.

The corresponding finite element approximation is extracting $[u_{0}(x), f_{0}(x)] : [0, T] \rightarrow V \times M$, to satisfy

$$\begin{align*}
(u_{0}(x), v) - (P, v) &= \langle (f, u_{0}(x)), v \rangle, \\
(P, q) + (\nabla u_{0}(x), q) + (\nabla u_{0}(x), q) &= (\nabla \xi_{x}, q), \quad \forall \phi_{z} \in M, \exists J
\end{align*}$$

(7)

\[ \text{III. ERROR ANALYSIS} \]

Let $u - u_{0} = u - f_{0} + f_{0} - f_{0} = \xi + q_{n}, \quad P - P_{0} = P - f_{0} + f_{0} - f_{0} = \rho + \theta$.

Theorem 1 assume $(\bar{u}, P)$ and $(\bar{u}, P)$ respectively are the solutions of the problem (6) and the problem (7), as $u, u_{0} \in H_{0}^{1}(\Omega)$, $P, P_{0} \in (H^{2}(\Omega))^{2}$, $u_{0} \in H_{0}^{1}(\Omega)$, we have

$$\begin{align*}
|u - u_{0}|^{2} &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(P, q) + (\nabla u_{0}(x), q) &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(8)

\[ \text{Proof} \] For the first equation and the second equation in (5) are acted by $\phi_{x}(\xi_{x}, v)$ and $\phi_{x}(\xi_{x}, v)$ on the two sides respectively, using Green formulation, we have

$$\begin{align*}
\int_{\Omega} \xi_{x} v d\Omega &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(P, q) + (\nabla u_{0}(x), q) &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(9)

According to (7), we have the following error equation

$$\begin{align*}
|u - u_{0}|^{2} &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(P, q) + (\nabla u_{0}(x), q) &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(10)

As $\epsilon \rightarrow 0$, it becomes

$$\begin{align*}
\int_{\Omega} \xi_{x} v d\Omega &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(P, q) + (\nabla u_{0}(x), q) &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(11)

On the basis of Schwartz inequality, we obtain

$$\int_{\Omega} (\nabla v, q) + (\nabla u_{0}(x), q) + (\nabla u_{0}(x), q) = - (\nabla \xi_{x}, q) = - (\nabla \xi_{x}, q) = \sum_{i=1}^{n} [\gamma(x_{i})]^{2}$$

(15)

As $\epsilon \rightarrow 0$, it becomes

$$\begin{align*}
\int_{\Omega} \xi_{x} v d\Omega &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(P, q) + (\nabla u_{0}(x), q) &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(16)

Integrate two side of (14), and pay attention to $\eta(X, 0) = 0$, we have

$$\begin{align*}
P_{x}P_{y} \leq c \int_{\Omega} |u_{0}|^{2} + \int_{\Omega} P_{x}P_{y} d\Omega
\end{align*}$$

(17)

In the first two equation of (12), respectively set $\nu^{*} = \eta$ and $\phi^{*} = \theta$, then add them, we obtain

$$\begin{align*}
\int_{\Omega} \xi_{x} v d\Omega &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}, \\
(P, q) + (\nabla u_{0}(x), q) &= \sum_{i=1}^{n} [\gamma(x_{i})]^{2}
\end{align*}$$

(18)

Using the properties of $J$ and Young inequality, the estimate of $A$ is

$$\begin{align*}
|A| \leq c \int_{\Omega} |u_{0}|^{2} + \int_{\Omega} P_{x}P_{y} d\Omega + c \epsilon |P_{x}P_{y}|^{2} + c \epsilon |P_{x}P_{y}|^{2}
\end{align*}$$

On the basis of interpolation theory, Schwartz inequality and Young inequality, we obtain
\[ \eta(X,0) = \eta(X,0) = 0, \]

For \( \eta = 0 \), similar to (14) and (15), we have

\[ \eta \leq \sum \left( b \cdot \nabla \chi \right) \eta \leq \sum \left( c \cdot \nabla \chi \right) \eta \leq + \int \left( c \cdot \nabla \chi \right) \eta. \]

On the basis of (2), we have

\[ \eta \left| \left| \eta \right| \right| \leq \left| \left| \chi \right| \right| + \left| \left| \left( \lambda \cdot \eta \right) \right| \right| \leq \left| \left| \left( \lambda \cdot \eta \right) \right| \right| \leq + \int \left( \lambda \cdot \eta \right). \]

Put them into (19), and thus the formulation (19) becomes

\[ \frac{d}{dt} \left( P_\eta \right) + P_\eta \leq c \cdot P_\eta \quad + c \cdot P_\eta \quad + \int \eta \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right). \]

Integrate two side, and pay attention to \( \eta(X,0) = \eta(X,0) = 0 \), we obtain

\[ \left( \int \eta \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right). \]

According to Gronwall lemma, we have

\[ \left( \int \eta \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right). \]

as \( \varepsilon \to 0 \), using Gronwall lemma, (15) and (17), we obtain

\[ \left( \int \eta \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right) \left( \left( \sum \left( \lambda \cdot \eta \right) \right) \right). \]

On the basis of triangle inequality and interpolation theory, we obtain (8) and (9) are true.

REFERENCES


