Fuzzy probability distribution with VaR constraint for portfolio selection

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Abstract

This work aims at comparing two models of fuzzy distribution: Normal and Laplace, whenever they are inside the context of possibilistic mean-variance model described by Li et al. in [6], where fuzzy Normal distribution is used. We propose to make a comparison using their model, but instead we apply fuzzy Laplace distribution. We also demonstrate the theorems which are necessary for the inclusion of these distributions to the model proposed by Li et al. So, we evaluate the behavior of this model when these distribution functions are changed and we also vary the VaR (Value at Risk). For financial analysts it is very important having other distributions as parameters, regarding the volatility of the stock market due to the behavior of financial market.

Keywords: Number Fuzzy, VaR, Portfolio selection, Risk

1. Introduction

The least complex and most natural way to represent the problem of optimal portfolio selection is a constrained optimization problem. The aim is to maximize or minimize an objective function (usually maximize returns or risk minimization) subject to constraints. However, the objective function and constraints are usually not simple functions. They often rely on more than one characteristic of each asset, and these characteristics are usually combinations of functions that are much more complex than a linear or quadratic function. So, finding a solution to this optimization problem requires more complex techniques.

For these and other reasons, many researchers seek models that can measure all of these variables. In this sense, Markowitz [7],[8] proposed a model for the mean-variance portfolio selection and probability theory associated to optimization techniques to model the performance of investment under uncertainty.

However, due to the complexity of financial systems, there are several situations where the input data are not precise but only fuzzy. Therefore,

the decision makers should not consider parameters (goals and constraints) using numbers or unique distribution functions, but instead they should use fuzzy numbers or fuzzy probability distribution functions (see, for example, Zdenek Zmeskal [16]). Recently, researchers investigated many fuzzy portfolio selection problems (see, e.g., Watada [13], Ramaswamy [9], Leon et al. [5], Wang and Zhu [12], Tanaka et al. [11] and Zhang et al. [17]). Carlsson and Fuller [2] introduced the notations of upper and lower possibilistic mean values, and introduced the notation of crisp possibilistic mean values and crisp possibilistic variance of continuous distributions. Zhang and Nie [18] extended the concepts of possibilistic mean and possibilistic variance proposed by Carlsson and Fuller [2], and introduced the concepts of upper and lower possibilistic variances and covariances of fuzzy numbers.

We propose to make a comparison using their model, but instead we apply fuzzy Laplace. We also demonstrate the theorems which are necessary for the inclusion of these distributions to the model proposed by Li et al. So, we evaluate the behavior of this model when these distributions functions are changed and we also vary the VaR (Value at Risk).

This paper is organized as follows. In section 2, we introduce the basic concepts of the possibilistic mean and variance of a fuzzy number. In section 3, we proposed a possibilistic portfolio model under constraints of VaR and risk-free investment. In section 4, we present a fuzzy normal distribution demonstrated in Li [6]. In section 5, we present fuzzy Laplace distribution. In Section 6, numerical examples are given to illustrate our effective proposed approaches. And finally, Section 7 presents our conclusions.

2. Possibilistic variance and mean value

In this section, we introduce some concepts that will be used in the next sections. A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\mathbb{R}$ with a normal, convex and continuous membership function of bounded support. The family of fuzzy number will be denoted by F . A γ -level set of fuzzy $\tilde{A}$ is defined by $[\tilde{A}]^\gamma = \{t \in \mathbb{R} | \tilde{A}(t) \geq \gamma\}$ if $\gamma > 0$ and $[\tilde{A}]^\gamma = \text{cl}\{t \in \mathbb{R} | \tilde{A}(t) > 0\}$ (the closure of the sup-

port of $\tilde{A}$) if $\gamma = 0$. It is well known that if $\tilde{A}$ is a fuzzy number then $[\tilde{A}]^\gamma$ is a compact subset of $\mathbb{R}$ for all $\gamma \in [0, 1]$ and $[\tilde{A}]^\gamma = [a_1(\gamma), a_2(\gamma)]$.

Carlsson and Fuller [2] defined the upper and lower possibilistic means of a fuzzy number $\tilde{A}$. First, we can define $\bar{M}(\tilde{A})$ as

$$\bar{M}(\tilde{A}) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma. \quad (1)$$

We can rewrite $\bar{M}(\tilde{A})$ as

$$\bar{M}(\tilde{A}) = \frac{1}{2} \left(\frac{\int_0^1 \gamma(a_1(\gamma))d\gamma}{\frac{1}{2}} + \frac{\int_0^1 \gamma(a_2(\gamma))d\gamma}{\frac{1}{2}} \right), \quad (2)$$

or

$$\bar{M}(\tilde{A}) = \frac{1}{2} \left(\frac{\int_0^1 \gamma(a_1(\gamma))d\gamma}{\int_0^1 \gamma d\gamma} + \frac{\int_0^1 \gamma(a_2(\gamma))d\gamma}{\int_0^1 \gamma d\gamma} \right). \quad (3)$$

Therefore,

$$\bar{M}(\tilde{A}) = \frac{\int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))/2d\gamma}{\int_0^1 \gamma d\gamma}. \quad (4)$$

It follows that $\bar{M}(\tilde{A})$ is nothing but the level-weighted average of the arithmetic means of all γ -level sets, that is, the weight of the arithmetic mean of $a_1(\gamma)$ and $a_2(\gamma)$ is just γ .

Lets look at the right-hand side of equation (4). The first quantity, denoted by $M_U(\tilde{A})$ can be reformulated as

$$M_U(\tilde{A}) = \frac{\int_0^1 \text{Pos}[\tilde{A} \leq a_1(\gamma)](a_1(\gamma))d\gamma}{\int_0^1 \text{Pos}[\tilde{A} \leq a_1(\gamma)]d\gamma}, \quad (5)$$

where Pos denotes possibility, i.e.

$$\text{Pos}[\tilde{A} \leq a_1(\gamma)] = \underbrace{\sup}_{u \leq a_1(\gamma)} A(u) = \gamma. \quad (6)$$

Similarly, we can denote $M_L(\tilde{A})$ as

$$M_L(\tilde{A}) = \frac{\int_0^1 \text{Pos}[\tilde{A} \geq a_2(\gamma)](a_2(\gamma))d\gamma}{\int_0^1 \text{Pos}[\tilde{A} \geq a_2(\gamma)]d\gamma} \quad (7)$$

where Pos denotes possibility, i.e.

$$\text{Pos}[\tilde{A} \geq a_2(\gamma)] = \underbrace{\sup}_{u \geq a_2(\gamma)} A(u) = \gamma \quad (8)$$

The possibilistic mean value of $\tilde{A}$ is the arithmetic mean of its lower and upper possibilistic mean values as follows:

$$\bar{M}(\tilde{A}) = \frac{M_U(\tilde{A}) + M_L(\tilde{A})}{2} \quad (9)$$

Similarly as shown for the upper and lower possibilistic means, Zhang and Nie [17] introduced the

upper and lower possibilistic variances and covariances fuzzy numbers. The upper and lower possibilistic variances of a fuzzy set $\tilde{A}$ is defined as:

$$\sigma_U^2(\tilde{A}) = 2 \int_0^1 \gamma(M_U(\tilde{A}) - a_2(\gamma))^2 d\gamma, \quad (10)$$

and

$$\sigma_L^2(\tilde{A}) = 2 \int_0^1 \gamma(M_L(\tilde{A}) - a_1(\gamma))^2 d\gamma. \quad (11)$$

respectively.

The upper and lower possibilistic covariances between fuzzy numbers, are defined, respectively, as:

$$\text{Cov}_U(\tilde{A}, \tilde{B}) = 2 \int_0^1 \gamma(M_U(\tilde{A}) - a_2(\gamma))(M_U(\tilde{B}) - b_2(\gamma))d\gamma, \quad (12)$$

and

$$\text{Cov}_L(\tilde{A}, \tilde{B}) = 2 \int_0^1 \gamma(M_L(\tilde{A}) - a_1(\gamma))(M_L(\tilde{B}) - b_1(\gamma))d\gamma. \quad (13)$$

The possibilistic variance of fuzzy number $\tilde{A}$ is defined as:

$$\overline{\sigma^2}(\tilde{A}) = \frac{\sigma_U^2(\tilde{A}) + \sigma_L^2(\tilde{A})}{2}. \quad (14)$$

The possibilistic covariance between the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as:

$$\overline{\text{Cov}}(\tilde{A}, \tilde{B}) = \frac{\text{Cov}_U(\tilde{A}, \tilde{B}) + \text{Cov}_L(\tilde{A}, \tilde{B})}{2}. \quad (15)$$

Lemma 2.1 [2] Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then for each $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\bar{M}(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}) = \lambda_1 \bar{M}(\tilde{A}) + \lambda_2 \bar{M}(\tilde{B}).$$

Lemma 2.2 [2] Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then for each $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned} \overline{\sigma^2}(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}) &= \lambda_1^2 \overline{\sigma^2}(\tilde{A}) + \lambda_2^2 \overline{\sigma^2}(\tilde{B}) \\ &\quad + 2|\lambda_1 \lambda_2| \overline{\text{Cov}}(\phi(\lambda_1) \tilde{A}, \phi(\lambda_2) \tilde{B}), \end{aligned}$$

where $\phi(x)$ is a sign function of $x \in \mathbb{R}$.

3. Portfolio Model under constraints of VaR and risk free investment

In order to define the model, it is necessary to make the following considerations. First, there are n risk assets and one risk-free asset for investment and the asset return rate $\tilde{\varphi}_i$ is a fuzzy number, $i = 1, 2, \dots, n$. x_i represents the proportion invested in assets i , and r_f is the risk-free asset return. From this, we can then define the return $\tilde{r}_p$ as

$$\tilde{r}_p = \sum_{i=1}^n x_i \tilde{\varphi}_i + r_f \left(1 - \sum_{i=1}^n x_i \right), \quad (16)$$

since $\tilde{\varphi}_i$ is a fuzzy membership $\tilde{r}_p$ is also a fuzzy number.

Consequently, the possibilistic mean of the portfolio return $\tilde{r}_p$ is

$$\overline{M}(\tilde{r}_p) = \sum_{i=1}^n x_i \frac{M_U(\tilde{\varphi}_i) + M_L(\tilde{\varphi}_i)}{2} + r_f \left(1 - \sum_{i=1}^n x_i \right). \quad (17)$$

According to Lemma 2.2, it is known that the possibilistic variance of $\tilde{r}_p$ is given by

$$\overline{\sigma^2} \left(\sum_{i=1}^n x_i \tilde{\varphi}_i \right) = \sum_{i=1}^n x_i^2 \overline{\sigma_{\varphi_i}^2} + 2 \sum_{i>j=1}^n x_i x_j \overline{Cov}(\tilde{\varphi}_i, \tilde{\varphi}_j). \quad (18)$$

A value at risk VaR constraint is imposed on our portfolio model, as follows. Replacing equation (6) and considering $a_1(\gamma) = VaR \leq 1 - \beta$ and β as confidence level, we obtain

$$Pos[\tilde{A} \leq VaR] = \sup_{t \geq VaR} A(t) \leq 1 - \beta. \quad (19)$$

Finally, we can define the possibilistic mean-variance model regarding the possibilistic mean as the portfolio return and the possibilistic variance as the portfolio risk. Under this structure, the possibilistic portfolio model under constraint of VaR and risk-free investment can be formulated as:

$$\begin{cases} \min & \overline{\sigma^2} = \sum_{i=1}^n x_i^2 \overline{\sigma_{\varphi_i}^2} + 2 \sum_{i>j=1}^n x_i x_j \overline{Cov}(\tilde{\varphi}_i, \tilde{\varphi}_j) \\ \text{s.t.} & \sum_{i=1}^n x_i \frac{M_U(\tilde{\varphi}_i) + M_L(\tilde{\varphi}_i)}{2} + r_f \left(1 - \sum_{i=1}^n x_i \right) \geq \tilde{r} \\ & pos(\tilde{\varphi}_i x_i \leq VaR) \leq 1 - \beta, \\ & \sum_{i=1}^n x_i \leq 1, \\ & 0 \leq l_i \leq x_i \leq u_i, i = 1, 2, \dots, n, \end{cases}$$

where pos denotes the measure of possibilistic, $\tilde{r}$ is the underestimated expected rate of return, l_i and u_i represent the lower bound and upper bound on investment in asset i , respectively, and VaR is defined as the value at risk by the β -confidence level.

The model shows that risk-averse investors wish not only to reach the expected rate of returns in their actual investment, but also to ensure that the maximum of their possible risk is lower than an expected loss.

4. Fuzzy Normal Distribution

According to Li [6] the rate of return on asset i is a fuzzy variable with fuzzy normal distribution expressed by $\tilde{\xi}_i \sim FN(\mu_i, \sigma_i)$, and its membership function is

$$A_{\tilde{\xi}_i}(t) = \exp\{ -[(t - \mu_i)/\sigma_i]^2 \}.$$

The γ -level set of $\tilde{\xi}$ is

$$[\tilde{\xi}_i]^\gamma = [\mu_i - \sigma_i \sqrt{\ln \gamma^{-1}}, \mu_i + \sigma_i \sqrt{\ln \gamma^{-1}}], \quad (20)$$

$\gamma \in (0, 1), i = 1, 2, \dots, n.$

Theorem 4.1 Assume that the return rates of assets are fuzzy variables with fuzzy normal distribution expressed as $\tilde{\varphi}_i \sim FN(\mu_i, \sigma_i), i=1,2,\dots,n$, then

$$\sum_{i=1}^n x_i \tilde{\varphi}_i \sim FN \left[\sum_{i=1}^n x_i \mu_i, \left(\frac{1}{2} - \frac{\pi}{8} \right) \left(\sum_{i=1}^n x_i \sigma_i \right)^2 \right], \quad (21)$$

where $x_i \geq 0, i = 1, 2, \dots, n.$

Proof

Refer to [6].

Moreover, Li [6] defines the possibilistic portfolio model under constraints of VaR and risk-free investments, whereas the variables are fuzzy with fuzzy Normal distribution. Thus:

$$\begin{cases} \min & \overline{\sigma^2} = \left(\frac{1}{2} - \frac{\pi}{8} \right) \left(\sum_{i=1}^n x_i^2 \sigma_i^2 + 2 \sum_{i>j=1}^n x_i x_j \sigma_i \sigma_j \right) \\ \text{s.t.} & \sum_{i=1}^n x_i (\mu_i - r_f) + r_f \geq \tilde{r} \\ & -(VaR - \sum_{i=1}^n x_i \mu_i)^2 \leq \ln(1 - \beta) \left(\frac{1}{2} - \frac{\pi}{8} \right) \left(\sum_{i=1}^n x_i \sigma_i \right)^2 \\ & \sum_{i=1}^n x_i \leq 1 \\ & 0 \leq l_i \leq x_i \leq u_i, i = 1, 2, \dots, n. \end{cases}$$

5. Fuzzy Laplace Distribution

Suppose that the return rate of asset i is a Laplace distribution fuzzy variable expressed as $\tilde{\varphi}_i \sim FL(\mu_i, \sigma_i)$, and its membership function is

$$\begin{aligned} A_{\tilde{\varphi}_i}(t/\mu_i, b) &= \frac{1}{2b} \exp\left(-\frac{|t - \mu_i|}{b}\right) \\ &= \frac{1}{2b} \begin{cases} \exp\left(\frac{-\mu_i + t}{b}\right) & \text{if } t < \mu_i \\ \exp\left(\frac{-t + \mu_i}{b}\right) & \text{if } t \geq \mu_i, \end{cases} \end{aligned}$$

where $\sigma_i^2 = 2b^2 \Rightarrow b = \frac{\sqrt{2}}{2} \sigma_i$.

The γ -level set of $\tilde{\varphi}_i$ is defined as

$$[\tilde{\varphi}_i]^\gamma = \left[\mu_i - \frac{\sqrt{2}}{2} \sigma_i \ln \sqrt{2} \sigma_i \gamma, \mu_i + \frac{\sqrt{2}}{2} \sigma_i \ln \sqrt{2} \sigma_i \gamma \right]. \quad (22)$$

Theorem 5.1 We assume that the rates of return of the assets are Laplace fuzzy distributions variables expressed as $\tilde{\varphi}_i \sim FL(\mu_i, \sigma_i), i = 1, 2, \dots, n$. Then

$$\sum_{i=1}^n x_i \tilde{\varphi}_i \sim FL \left(\sum_{i=1}^n x_i \mu_i, \frac{1}{8} \left(\sum_{i=1}^n x_i \sigma_i \right)^2 \right), \quad (23)$$

where $x_i \geq 0, i = 1, 2, \dots, n.$

Proof

According to Lemma 2.1, the possibilistic mean value of $\sum_{i=1}^n x_i \tilde{\varphi}_i$ can be calculated by

$$\overline{M}\left(\sum_{i=1}^n x_i \tilde{\varphi}_i\right) = \sum_{i=1}^n x_i \overline{M}(\tilde{\varphi}_i) = \sum_{i=1}^n x_i \mu_i.$$

From equations (5), (7) and (21), it can be deduced that

$$\begin{aligned} M_U(\tilde{\varphi}_i) &= 2 \int_0^1 \gamma \left(\mu_i + \frac{\sqrt{2}}{2} \sigma_i \ln \sqrt{2} \sigma_i \gamma \right) d\gamma \\ &= \mu_i + \sqrt{2} \sigma_i \int_0^1 \gamma \ln \sqrt{2} \sigma_i \gamma d\gamma \\ &= \mu_i + \frac{\sqrt{2}}{2} \sigma_i \left(\ln \sqrt{2} \sigma_i - \frac{1}{2} \right). \end{aligned} \quad (24)$$

and

$$\begin{aligned} M_L(\tilde{\varphi}_i) &= 2 \int_0^1 \gamma \left(\mu_i - \frac{\sqrt{2}}{2} \sigma_i \ln \sqrt{2} \sigma_i \gamma \right) d\gamma \\ &= \mu_i - \frac{\sqrt{2}}{2} \sigma_i \left(\ln \sqrt{2} \sigma_i - \frac{1}{2} \right). \end{aligned} \quad (25)$$

In the same way, the following results can be obtained:

$$\begin{aligned} \sigma_U^2 &= 2 \int_0^1 \gamma \left[M_U(\tilde{\varphi}_i) - a_2(\gamma) \right]^2 d\gamma \\ &= \frac{1}{8} \sigma_i^2, \end{aligned}$$

and

$$\begin{aligned} \sigma_L^2 &= 2 \int_0^1 \gamma \left[M_L(\tilde{\varphi}_i) - a_2(\gamma) \right]^2 d\gamma \\ &= \frac{1}{8} \sigma_i^2. \end{aligned}$$

Thus, the possibilistic variance can be written as

$$\overline{\sigma_{\tilde{\varphi}_i}^2} = \frac{\sigma_U^2 + \sigma_L^2}{2} = \frac{1}{8} \sigma_i^2. \quad (26)$$

Furthermore, according to equations (12) and (13), the upper and lower possibilistic covariances are given by

$$\begin{aligned} Cov_U(\tilde{\varphi}_i, \tilde{\varphi}_j) &= 2 \int_0^1 \gamma \left[M_U(\tilde{\varphi}_i) - a_2(\gamma) \right] \\ &\quad \left[M_U(\tilde{\varphi}_j) - b_2(\gamma) \right] d\gamma = \frac{1}{8} \sigma_i \sigma_j, \\ Cov_L(\tilde{\varphi}_i, \tilde{\varphi}_j) &= 2 \int_0^1 \gamma \left[M_L(\tilde{\varphi}_i) - a_1(\gamma) \right] \\ &\quad \left[M_L(\tilde{\varphi}_j) - b_1(\gamma) \right] d\gamma = \frac{1}{8} \sigma_i \sigma_j, \end{aligned}$$

The possibilistic covariance is

$$\begin{aligned} \overline{Cov}(\tilde{\varphi}_i, \tilde{\varphi}_j) &= \frac{Cov_U(\tilde{\varphi}_i, \tilde{\varphi}_j) + Cov_L(\tilde{\varphi}_i, \tilde{\varphi}_j)}{2} \\ &= \frac{1}{8} \sigma_i \sigma_j. \end{aligned} \quad (27)$$

According to Lemma 2.2 and $(x_i \geq 0)$, we can compute the possibilistic variance of $\sum_{i=1}^n x_i \tilde{\varphi}_i$

$$\begin{aligned} \overline{\sigma_{\sum_{i=1}^n x_i \tilde{\varphi}_i}^2} &= \sum_{i=1}^n x_i^2 \overline{\sigma_{\tilde{\varphi}_i}^2} + 2 \sum_{i=1}^n x_i x_j \overline{Cov}(\tilde{\varphi}_i, \tilde{\varphi}_j) \\ &= \sum_{i=1}^n \frac{1}{8} x_i^2 \sigma_i^2 + 2 \sum_{i=1}^n \frac{1}{8} x_i x_j \sigma_i \sigma_j \\ &= \frac{1}{8} \left(\sum_{i=1}^n x_i \sigma_i \right)^2. \end{aligned}$$

So the proof of the theorem is complete. $\square$

According to Theorem 5.1 the membership function of $\sum_{i=1}^n x_i \tilde{\varphi}_i$ is defined by:

$$A(t) = \frac{1}{2b} \exp \left\{ - \frac{|t - (\sum_{i=1}^n x_i \mu_i)|}{b} \right\},$$

where

$$b = \frac{\sqrt{2}}{2} \sigma \Rightarrow b = \frac{\sqrt{2}}{2} \left(\frac{1}{8} \left(\sum_{i=1}^n x_i \sigma_i \right)^2 \right).$$

So

$$\begin{aligned} pos \left(\sum_{i=1}^n \varphi_i x_i \leq VaR \right) &= \sup_{t \leq VaR} \left\{ \frac{1}{2b} \exp \left\{ - \frac{\left| t - \left(\sum_{i=1}^n x_i \mu_i \right) \right|}{b} \right\} \right\}, \end{aligned} \quad (28)$$

$$\begin{aligned} &= \frac{1}{2b} \exp \left\{ - \frac{|VaR - (\sum_{i=1}^n x_i \mu_i)|}{b} \right\} \\ &= \begin{cases} \frac{1}{2b} \exp \left\{ - \frac{\sum_{i=1}^n x_i \mu_i - VaR}{b} \right\} & \text{if } VaR < \sum_{i=1}^n x_i \mu_i \\ \frac{1}{2b} \exp \left\{ - \frac{VaR - \sum_{i=1}^n x_i \mu_i}{b} \right\} & \text{if } VaR \geq \sum_{i=1}^n x_i \mu_i \end{cases} \end{aligned} \quad (29)$$

According to equations (20) and (29), we can obtain

$$\left(VaR - \sum_{i=1}^n x_i \mu_i \right) \leq b \ln [2b(1 - \beta)] \text{ if } VaR < \sum_{i=1}^n x_i \mu_i$$

$$\left(\sum_{i=1}^n x_i \mu_i - VaR \right) \leq b \ln [2b(1 - \beta)] \text{ if } VaR \geq \sum_{i=1}^n x_i \mu_i.$$

It is known from equations (16), (24) and (17), that when the return rate of assets are Laplace distribution fuzzy variables, the upper and lower possibilistic means of $\tilde{r}_p$ are given by:

$$M_U(\tilde{r}_p) = \sum_{i=1}^n \left(\mu + \frac{\sqrt{2}}{2} \sigma \left(\ln \sqrt{2} \sigma - \frac{1}{2} \right) - r_f \right) x_i + r_f$$

$$M_L(\tilde{r}_p) = \sum_{i=1}^n \left(\mu - \frac{\sqrt{2}}{2} \sigma \left(\ln \sqrt{2} \sigma - \frac{1}{2} \right) - r_f \right) x_i + r_f$$

Thus, the possibilistic mean of $\tilde{r}_p$ is written as

$$\bar{M}(\tilde{r}_p) = \frac{M_U(\tilde{r}_p) + M_L(\tilde{r}_p)}{2} = \sum_{i=1}^n (\mu_i - r_f) x_i + r_f.$$

Moreover, for a Laplace fuzzy variable distribution, the portfolio model of VaR restrictions and risk free investment can be formulated as:

$$\left\{ \begin{array}{l} \min \quad \bar{\sigma}^2 = \frac{1}{8} \left(\sum_{i=1}^n x_i^2 \sigma_i^2 + 2 \sum_{i>j=1}^n x_i x_j \sigma_i \sigma_j \right) \\ s.t. \quad \sum_{i=1}^n x_i (\mu_i - r_f) + r_f \geq \tilde{r} \\ (VaR - \sum_{i=1}^n x_i \mu_i) \leq b \ln [2b(1 - \beta)] p / VaR < \sum_{i=1}^n x_i \mu_i, \\ (\sum_{i=1}^n x_i \mu_i - VaR) \leq b \ln [2b(1 - \beta)] p / VaR \geq \sum_{i=1}^n x_i \mu_i, \\ \text{where } b = \frac{\sqrt{2}}{16} \left(\sum_{i=1}^n x_i \sigma_i \right)^2 \\ 0 \leq l_i \leq x_i \leq u_i, i = 1, 2, \dots, n, \end{array} \right.$$

6. Numerical Example

In order to compare the results using the three models of fuzzy distributions, namely, Normal and Laplace, whenever they are in the context of possibilistic mean-variance model described by Li et al, we selected three investment funds from Bradesco, using data from January to December of 2014. In a next step, we generated a database with 5290 possibilities x_i , $i = 1, 2, \dots, n$, representing the proportion invested in assets i , namely, $0 \leq l_i \leq x_i \leq u_i$, $i = 1, 2, \dots, n$. For our work, we will consider $n = 3$ assets and return rate for each asset $\tilde{\varphi}_i \sim FN(\mu_i, \sigma_i)$ and $\tilde{\varphi}_i \sim FL(\mu_i, \sigma_i)$, is calculated from the frequency distribution of months returns of the three assets, see Table 1.

Then we will analyze the results for each distribution function where $VaR = 0.02$ and $\beta = 0.9$ and the return of risk-free asset considered is 0.72%.

Table 1: The possibilistic distributions of returns of three assets

	Assets		
	1	2	3
μ	0.7452	0.7035	0.8087
σ	0.0319	0.0294	0.0278

6.1. Result for fuzzy Normal distribution

In Figure 01, it was considered the return rate between 0.1% and 10% to the maximum risk rate of 10%. We noted that the 3rd possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 10% in assets 03 and 88% risk-free.

Figure 1: Result for Normal distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 02 we considered the return rate between 10% and 20% to the maximum risk rate of 10%. We noted that the 7th possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 18.8% in assets 03 and 79.2% risk-free.

Figure 2: Result for Normal distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 03 we considered the return rate between 20% and 30% to the maximum risk rate of 10%. We noted that the 10th possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 27.7% in assets 03 and 70.3% risk-free.

Figure 3: Result for Normal distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 04 we considered the return rate between 30% and 40% to the maximum risk rate of 20%. We noted that the 8th possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 45.5% in assets 03 and 52.5% risk-free.

Figure 4: Result for Normal distribution with $VaR = 0.02$ and $\beta = 0.9$.

6.2. Result for fuzzy Laplace distribution

In Figure 05 we considered the return rate between 0.1% and 10% to the maximum risk rate of 10%. We noted that the 3rd possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 9.9% in assets 03 and 88.1% risk-free.

Figure 5: Result for Laplace distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 06 we considered the return rate between 10% and 20% to the maximum risk rate of 10%. We noted that the 7th possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 18.8% in assets 03 and 79.2% risk-free.

Figure 6: Result for Laplace distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 07 we considered the return rate between 20% and 30% to the maximum risk rate of 10%. We noted that the 6th possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 27.7% in assets 03 and 70.3% risk-free.

Figure 7: Result for Laplace distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 08 we considered the return rate between 30% and 40% to the maximum risk rate of 20%. We noted that the 4th possibility is one of the best choices for the investor, and investments would be distributed as follows: 1% on assets 01, 1% on assets 02, 36.6% in assets 03 and 61.4% risk-free.

Figure 8: Result for Laplace distribution with $VaR = 0.02$ and $\beta = 0.9$.

In Figure 09 we presented the difference between the risk rates calculated by Normal distribution and Laplace distribution. Negative values indicate that the risk ratios calculated, when we use Laplace distribution is lower, but it is a small difference.

Figure 9: Result for Laplace distribution with $VaR = 0.02$ and $\beta = 0.9$.

7. Conclusion

This work compared models of fuzzy distribution, namely, Normal and Laplace, whenever they are inside the context of possibilistic mean-variance model described by Li et al. in [6], where fuzzy Normal distribution is used. We proposed to make a comparison using their model, but instead we applied fuzzy Laplace distribution. We also demonstrated the theorems which were necessary for the inclusion of these distributions to the model proposed by Li et al. So, we evaluated the behavior of this model when these distribution functions are changed and we also varied the VaR (Value at Risk). It is well known the importance of having other distributions as parameters for financial analysts, due the volatility of the stock market as well as the behavior of financial market.

We also observed the importance of working with several fuzzy probability distributions, as demonstrated by the significant variation in the return and risk rates. Besides, we demonstrated that for the model proposed by Li et al. the fuzzy Normal distribution and fuzzy Laplace distribution are the most appropriate ones.

Therefore, risk-averse investors were given the possibility of a better evaluation, i.e, deciding the best way to distribute their funds in assets investment.

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