

Łukasiewicz-like triangular subnorms

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Abstract

The paper defines and studies Łukasiewicz-like triangular subnorms, i.e., triangular subnorms such that their level sets are obtained as unions of the level sets of the Łukasiewicz triangular norm. As a result, the paper gives a characterization of these operations.

Keywords: Positively cancellative, commutative semigroup, triangular subnorm, weakly cancellative.

1. Introduction

Triangular subnorms, introduced by Jenei [Jen01], are known as a weakening of the notion of a triangular norm [AFS, KMP, SchSk1] by withdrawing the axiom of a neutral element. The notion was motivated by observation that an ordinal sum [KMP] of left-continuous triangular subnorms is a left-continuous triangular norm. Recall that left-continuous triangular norms play a crucial role in the *monoidal triangular norm based logic* (abbreviated by *MTL*) [EstGod] as they form their standard semantics. The full characterization of triangular subnorms is still missing; even the continuous case is still not clear although an effort has been made in this direction already [Mes04, Mes05].

In this paper we focus on, as we call them, Łukasiewicz-like triangular subnorms, i.e., those that can be obtained by a simple transformation of the Łukasiewicz triangular norm, here denoted by \odot , and therefore their level set plot is identical with the level set plot of \odot . Our aim is to give a characterization of these triangular subnorms. As these operations are given as compositions of a real function g and \odot , the characterization will be performed by giving a sufficient and necessary conditions on g .

2. Preliminaries

Let us start with the definitions of some basic notions.

Definition 2.1 A triangular subnorm (shortly t-subnorm) is a binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ such that, for every $x, y, z \in [0, 1]$,

1. $x \leq y$ implies $x * z \leq y * z$,
2. $x * y = y * x$,

$$3. (x * y) * z = x * (y * z),$$

$$4. x * y \leq x \text{ and } x * y \leq y.$$

Definition 2.2 A triangular norm (shortly t-norm) is a binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ such that, for every $x, y, z \in [0, 1]$,

$$1. x \leq y \text{ implies } x * z \leq y * z,$$

$$2. x * y = y * x,$$

$$3. (x * y) * z = x * (y * z),$$

$$4. x * 1 = x.$$

It can be seen that t-norms are exactly those t-subnorms where $x * 1 = x$ holds for every $x \in [0, 1]$. A t-subnorm is called *proper* if it is not a t-norm. In the other way round, for every t-subnorm $*$, one can easily construct a related t-norm \otimes simply by redefining its margins:

$$x \otimes y = \begin{cases} x * y & \text{if } (x, y) \in [0, 1]^2, \\ \min\{x, y\} & \text{otherwise.} \end{cases} \quad (1)$$

As an example of a t-norm, let us mention the Łukasiewicz t-norm given, for every $x, y \in [0, 1]$, by

$$x \odot y = \max\{0, x + y - 1\}.$$

Recall that this operation plays a prominent role in MV-algebras [CiOtMu].

Definition 2.3 A t-subnorm $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be cancellative if, for every $x, y, z \in [0, 1]$ such that $x, y, z > 0$,

$$x * y = x * z \text{ implies } y = z.$$

Definition 2.4 A t-subnorm $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be positively cancellative if, for every $x, y, z \in [0, 1]$,

$$x * y = x * z > 0 \text{ implies } y = z.$$

As we can observe, the cancellative t-subnorm are just special cases of positively cancellative ones.

Remark 2.5 It can be observed easily that the t-norm constructed from a cancellative resp. positively cancellative t-subnorm according to (1) is cancellative resp. positively cancellative as well.

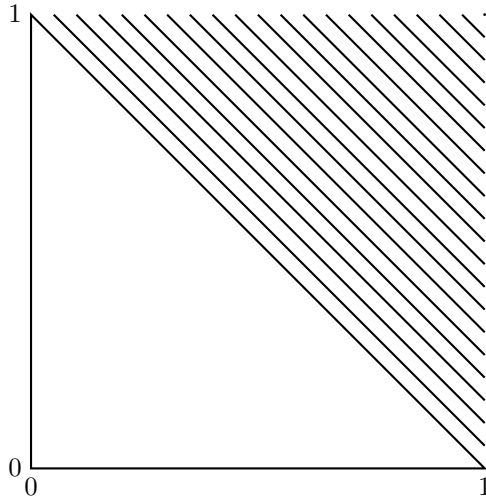


Figure 1: Level set plot of the Łukasiewicz triangular norm.

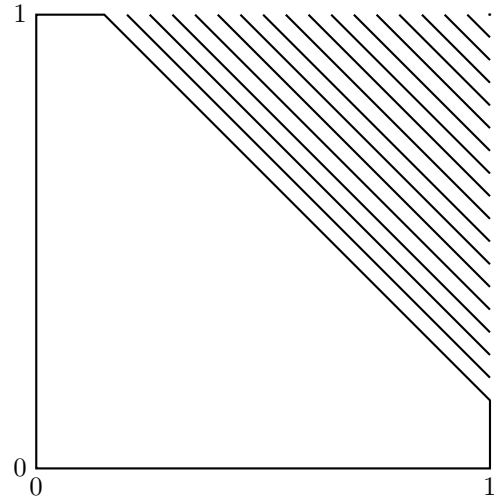


Figure 3: Level set plot of a positively cancellative Łukasiewicz-like triangular subnorm.

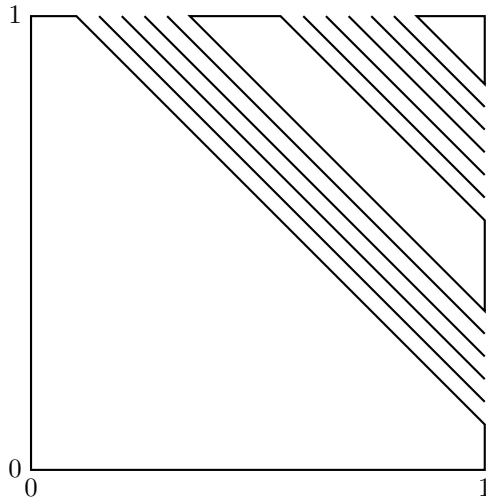


Figure 2: Level set plot of a general Łukasiewicz-like triangular subnorm.

Remark 2.6 Our notion of positive cancellativity is identical with the notion of weak cancellativity introduced by Montagna, Noguera, and Horčík when studying weakly cancellative MTL logics [MoNoHo]. The notion of weak cancellativity has been, however, historically defined also in different meanings [CKMN, Sch]. This is why we have decided to stick to the notion of positive cancellativity as we find it, in the case of triangular subnorms, more descriptive. We are aware that this property is also known as the conditional cancellativity [KMP].

3. Łukasiewicz-like t-subnorms

Let us start with the following two definitions.

Definition 3.1 A z -level set of a binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is the subset L_z^* of $[0, 1] \times$

$[0, 1]$ given by

$$L_z^* = \{(x, y) \in [0, 1] \times [0, 1] \mid x * y = z\}.$$

Definition 3.2 A t -subnorm is said to be Łukasiewicz-like if it can be expressed as a composition $g \circ \odot$ where g is a real function $g: [0, 1] \rightarrow [0, 1]$ and \odot is the Łukasiewicz t -norm.

As we can see, Łukasiewicz-like t -subnorms are defined such that their systems of level sets correspond with the system of level sets of the Łukasiewicz t -norms. In the case of the Łukasiewicz t -norm, the 0-level set is the triangle with its vertices in $(0, 0)$, $(1, 0)$, and $(0, 1)$; the other level sets have the shape of a line segment that form the angle of 45° with both the axis. This case is illustrated in Figure 1. On the other hand, we obtain the level set system of a Łukasiewicz-like t -subnorm simply by unifying some of the level sets of the Łukasiewicz t -norm. Due to the monotonicity of t -subnorms, the resulting sets are always connected and convex. This case is illustrated in Figure 2. To obtain all the positively cancellative Łukasiewicz-like t -subnorms, we follow the same process but we unify the level sets only with the 0-level set. This case is illustrated in Figure 3.

In the case of t -norms, the level set plot are in a 1-1 correspondence with the operations they describe. This is due to the fact that t -norms possess the neutral element 1. Thanks to this, looking at a level set plot, we can assign to each level set its level simply by determining the point where the level set intersect the axis given by the point $(1, 1)$. This is, however, not the case of t -subnorms as they, in general, do not possess a neutral element. Thus, multiple t -subnorms can share one and the same level set plot. Therefore, to determine whether a level set plot describes a t -subnorm, the key is to determine the levels of the level sets. As a result

of this paper, we are going to introduce a necessary and sufficient condition of the levels of the level set system of a Łukasiewicz-like t-subnorm.

For a given operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$, we define a relation \approx on $[0, 1] \times [0, 1]$; two points of $[0, 1] \times [0, 1]$ are in the relation if they belong to the same level set of $*$, i.e.,

$$(a, b) \approx (c, d) \Leftrightarrow a * b = c * d.$$

This relation is an equivalence; we call it the *level equivalence* of $*$ and its equivalence classes are the level sets of $*$.

We stress out here one important feature of the level set systems of the operations of the type $g \circ \odot$ (cf. Definition 3.2). Remark that this set is strictly larger than the set of Łukasiewicz-like t-subnorms. Since level sets of such an operation are given as unions of line segments with the border vertices of the form $(t, 0)$ and $(0, t)$, where $t \in [0, 1]$, we obtain that two points $(a, b), (c, d) \in [0, 1] \times [0, 1]$ belong to the same level set of the t-subnorm if $a + b = c + d$.

Lemma 3.3 *Let $*$ be a binary operation on $[0, 1]$ such that $*$ = $g \circ \odot$ where g : $[0, 1] \rightarrow [0, 1]$ and \odot is the Łukasiewicz t-norm.*

$$\begin{aligned} \text{If } a + b = c + d \quad \text{then } (a, b) \approx (c, d) \\ \text{and } a * b = c * d. \end{aligned}$$

4. Representation of Łukasiewicz-like t-subnorms

A prominent, one-parameter family of Łukasiewicz-like t-subnorms is formed by the operations

$$*_{*w}: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

with

$$x *_{*w} y = \max\{0, x + y - 1 - w\}$$

for every $x, y \in [0, 1]$ where the parameter w is allowed to range over the unit interval. It is straightforward to check that $*_{*w} = g_w \circ \odot$ where \odot is the Łukasiewicz-like t-norm and g_w is the mapping

$$z \mapsto \max\{z - w, 0\}.$$

In the sequel we give a full characterization of the continuous, positively cancellative Łukasiewicz-like t-subnorms in terms of the function g .

We commence with several necessary conditions.

Lemma 4.1 *Let g : $[0, 1] \rightarrow [0, 1]$ be a transformation of the unit interval. If the composition $*$ = $g \circ \odot$ is a t-subnorm, then*

$$g(z) \leq z$$

for every $z \in [0, 1]$.

Proof Suppose there is $z \in [0, 1]$ such that $g(z) > z$. Then $z * 1 = g(z \odot 1) = g(z) > z$, i.e., $z * 1 > z$ which is in a contradiction with Item 4 of Definition 2.1. ■

Lemma 4.2 *Let g : $[0, 1] \rightarrow [0, 1]$ be a transformation of the unit interval. If the composition $*$ = $g \circ \odot$ is a continuous, positively cancellative t-subnorm, then g is continuous and, moreover, there exists $v \in [0, 1[$ such that*

- the restriction of g on $[0, v]$ is the constant 0,
- the restriction of g on $]v, 1]$ is a positive, strictly increasing function.

Proof Notice first that

$$z * 1 = 1 * z = g(z)$$

holds for every $z \in [0, 1]$. Since $*$ is continuous, non-negative, and non-decreasing, so is g . Let us denote

$$v = \sup\{z \in [0, 1] \mid z * 1 = 1 * z = g(z) = 0\}.$$

By non-decreasingness and non-negativity of g it is clear that the restriction of g on $[0, v]$ is a constant function equal to 0. From the definition of v and from the non-decreasingness of g it also follows that the restriction of g on $]v, 1]$ assumes positive values only. Finally, by assumption of the positive cancellativity

$$g(y) = 1 * y = 1 * z = g(z) > 0$$

implies $y = z$ for every $y, z \in]v, 1]$. In other words, the positive part of g , which is its restriction on $]v, 1]$, is injective and, hence, strictly increasing. ■

Lemma 4.3 . *Let g : $[0, 1] \rightarrow [0, 1]$ be a transformation of the unit interval. Denote*

$$v = \sup\{z \in [0, 1] \mid g(z) = 0\}, \quad (2)$$

$$w = 1 - g(1), \quad (3)$$

$$s = \min\{v + w, 1\}. \quad (4)$$

If the operation $$ = $g \circ \odot$ is a continuous, positively cancellative t-subnorm then the restriction of g on $[s, 1]$ is given by the expression*

$$g(z) = z - w. \quad (5)$$

Proof Since $*$ is associative, we have

$$(x * y) * z = x * (y * z)$$

for every $x, y, z \in [0, 1]$. This equality can be rewritten, according to the definition of $*$, to

$$g(g(x \odot y) \odot z) = g(x \odot g(y \odot z)).$$

For $x = y = 1$ we obtain

$$\begin{aligned} g(g(1 \odot 1) \odot z) &= g(1 \odot g(1 \odot z)), \\ g(g(1) \odot z) &= g(g(z)) \end{aligned}$$

for every $z \in [0, 1]$. According to Lemma 4.2, the positive part of g is injective. Therefore, assuming that both sides of the equation are positive, we can cancel the outer g in the equation; this happens if, and only if, the arguments of the outer g take their values from the interval $]v, 1]$. It is easy to check that the left-hand side argument, $g(1) \odot z$, lies in $]v, 1]$ if, and only if, $z > s$.

Thus, according to the definition of the Łukasiewicz t-norm, we have

$$\begin{aligned} g(1) \odot z &= g(z), \\ \max\{0, g(1) + z - 1\} &= g(z), \\ \max\{0, z - w\} &= g(z) \end{aligned}$$

whenever $z > s$. Furthermore, since $g(1) + z - 1 > 0$ whenever $z > s$, the truncation by 0 in the latter equation can be omitted. ■

Corollary 4.4 *Let $g: [0, 1] \rightarrow [0, 1]$ be a transformation of the unit interval. Denote the values $v, w, s \in [0, 1]$ according to (2), (3), and (4), respectively. If the operation $* = g \circ \odot$ is a continuous, positively cancellative t-subnorm then we have:*

$$\begin{aligned} \text{if } s < 1 \quad \text{then } g(s) &= v, \\ \text{if } s = 1 \quad \text{then } g(s) &\leq v. \end{aligned}$$

Proof If $s < 1$ then, according to (4),

$$s = v + w$$

and, according to (5),

$$g(z) = z - w$$

for every $z \in [s, 1]$. Therefore $g(s) = v$.

Suppose that $s = 1$ and $g(s) = g(1) > v$. Then

$$\begin{aligned} g(1) &> v, \\ 1 - g(1) &< 1 - v, \\ w &< 1 - v, \\ v + w &< 1, \end{aligned}$$

and thus, according to (4),

$$s = v + w < 1$$

which is a contradiction. ■

We continue with the main result of the paper. The function g described in the theorem is illustrated in Figure 4.

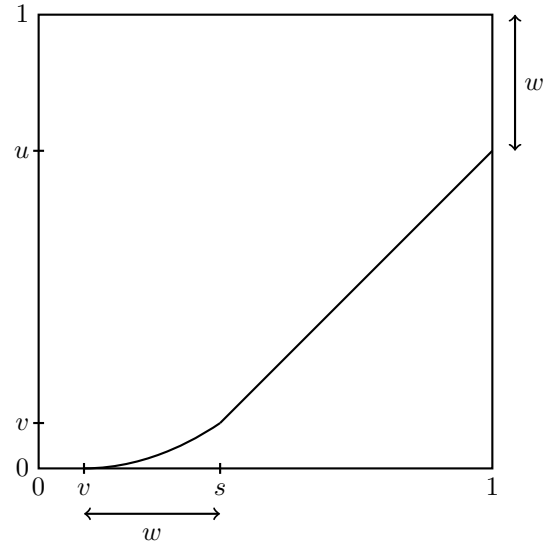


Figure 4: Illustration of a function $g: [0, 1] \rightarrow [0, 1]$ such that $* = g \circ \odot$ is a continuous, positively cancellative Łukasiewicz-like triangular subnorm.

Theorem 4.5 *Let g be a transformation of the unit interval. Denote*

$$\begin{aligned} v &= \sup\{z \in [0, 1] \mid g(z) = 0\}, \\ w &= 1 - g(1), \\ s &= \min\{v + w, 1\}. \end{aligned}$$

The composition $ = g \circ \odot$, where \odot denotes the Łukasiewicz t-norm, is a continuous, positively cancellative t-subnorm if, and only if,*

1. $g(z) \leq z$ for every $z \in [0, 1]$,
2. g is continuous and non-decreasing,
3. the restriction of g on $[0, v]$ is the constant 0,
4. the restriction of g on $]v, s]$ is a positive, strictly increasing function,
5. the restriction of g on $]s, 1]$ is given by the expression

$$g(z) = z - w.$$

Proof The “only if” part of the proof follows from the introduced lemmas. Item 1 follows from Lemma 4.1. Item 2, Item 3, and Item 4 follow all from Lemma 4.2. Item 5 follows from Lemma 4.3.

To prove the “if” part, we first show that $* = g \circ \odot$ is, actually, a t-subnorm. Thus we need to prove the four items of Definition 2.1.

1. Both \odot and g are non-decreasing functions. Therefore also $* = g \circ \odot$ is non-decreasing.
2. Commutativity of $*$ follows directly from the commutativity of \odot .

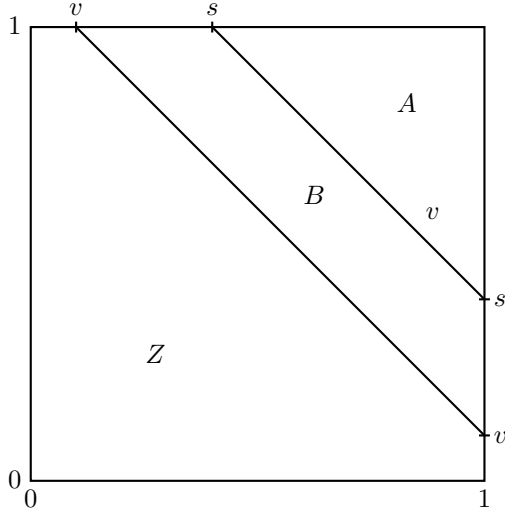


Figure 5: Division of the unit square into three subsets according to the coordinates v and s . Notice that the line segment from $(v, 1)$ to $(1, v)$ is actually a border of the 0-level set and that the line segment from $(s, 1)$ to $(1, s)$ is actually the v -level set.

3. Suppose first that $s = 1$. In this case we always have, due to Corollary 4.4,

$$\begin{aligned} x * y &\leq v, \\ y * z &\leq v \end{aligned}$$

and therefore

$$\begin{aligned} (x * y) * z &= 0, \\ x * (y * z) &= 0. \end{aligned}$$

To prove the associativity of $*$ for the case when $s < 1$ we divide the unit square into three subsets $A, B, Z \subseteq [0, 1] \times [0, 1]$ as depicted in Figure 5:

$$\begin{aligned} A &= \{(x, y) \in [0, 1] \times [0, 1] \mid x + y > 1 + s\}, \\ B &= \{(x, y) \in [0, 1] \times [0, 1] \mid \\ &\quad 1 + v < x + y \leq 1 + s\}, \\ Z &= \{(x, y) \in [0, 1] \times [0, 1] \mid x + y \leq 1 + v\}. \end{aligned}$$

The following cases may happen.

- (a) Suppose that $(x, y), (y, z) \in A$. Then

$$\begin{aligned} (x * y) * z &= (x + y - 1 - w) * z, \\ x * (y * z) &= x * (y + z - 1 - w). \end{aligned}$$

These two expressions are, however, equal invoking Lemma 3.3.

- (b) Suppose that $(x, y) \in A$ and $(y, z) \in B$. Then

$$\begin{aligned} (x * y) * z &= (x + y - 1 - w) * z, \\ &= g(x + y + z - 2 - w). \end{aligned}$$

Due to $(y, z) \in B$ we have $y + z < 1 + s$. Thus, invoking (4),

$$\begin{aligned} x + y + z - 2 - w &< x + 1 + s - 2 - w, \\ x + y + z - 2 - w &< x + s - w - 1, \\ x + y + z - 2 - w &< x + v - 1, \\ x + y + z - 2 - w &< v - (1 - x), \\ x + y + z - 2 - w &\leq v \end{aligned}$$

since $1 - x \geq 0$. Therefore

$$g(x + y + z - 2 - w) = (x * y) * z = 0.$$

Further, observe that $y + z < 1 + s$ implies $y * z \leq v$. Therefore

$$x * (y * z) = 0.$$

(The case when $(x, y) \in B$ and $(y, z) \in A$ is analogous.)

- (c) Suppose that $(x, y) \in Z$. Then

$$x * y = 0 \quad (6)$$

and, therefore, $(x * y) * z = 0$. From the monotonicity of $*$ and from (6) we have $y * z \leq y$ and, thus, $x * (y * z) = 0$.

(The case when $(y, z) \in Z$ is analogous.)

4. This part follows from Assumption 1 of the theorem.

Finally, the continuity and positive cancellativity of $*$ follows from the continuity of g and from the fact that g is strictly increasing on $]s, 1]$. ■

5. Concluding remark

A representation theorem for continuous, positively cancellative Łukasiewicz-like t-subnorms has been given. It is our conjecture and intention for the further research to show that these t-subnorms are prototypical examples of all continuous, positively cancellative t-subnorms. Clearly, if $*$ is a positively cancellative Łukasiewicz-like t-subnorm then, for any increasing bijection $f: [0, 1] \rightarrow [0, 1]$, the operation $\bullet: [0, 1] \times [0, 1] \rightarrow [0, 1]$ given, for every $x, y \in [0, 1]$, by

$$x \bullet y = f^{-1}(f(x) * f(y))$$

is a continuous positively cancellative t-subnorm.

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