N-contrapositivisation of fuzzy implication functions

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Abstract

The law of contraposition with respect to a negation (usually strong) is one of the most studied properties in the theory of fuzzy implication functions. We already know some methods for modifying an implication with the aim that the new implication satisfies this property, these methods are called contrapositivisation. In this paper we present two new methods of contrapositivisation with respect to any strong negation and we study their properties. Along this study we will see that these new methods not only preserve the usual properties preserved by the already known methods, but they also have some additional property.

Keywords: Fuzzy implication, contraposition, strong negation, contrapositivisation.

1. Introduction

Fuzzy implication functions are key operators in fuzzy control and approximate reasoning, and also in all the fields where these theories apply. For this reason many authors have dealt with fuzzy implications both from a theoretical and from an applied point of view; thus fuzzy implications have become a research field, as we can see in the review paper [16] and in the fact that some books have appeared on this topic (see [4, 3]).

Due to the great number of applications they have, there exist many different models of fuzzy implication functions, depending on the particular problem they have to model. The most accepted definition of a fuzzy implication function is very general, and it only requires the monotonicities and the condition that it coincides with the classical implication at 0 and 1. Thus many of the studies on fuzzy implications deal with additional properties that could be desirable in each particular case. Most of these properties come from classical logic tautologies that become functional equations when they are translated into fuzzy logic. The solutions of these functional equations give different types of implications that satisfy the required algebraic properties.

In [18] it is indicated that the characterizations through algebraic properties are essential for understanding the behaviour of the different models of fuzzy implications. Thus, for example, the (S, N)-implications (except when the negation N is not continuous) and the R-implications are completely characterized (see [2] and [9] respectively), but the QL and D-operations are only characterized in some particular cases (see [15]).

One of the most studied properties is the so-called *law of contraposition* of an implication I with respect to a negation N, expressed as

$$I(N(y), N(x)) = I(x, y) \quad \text{for all } x, y \in [0, 1]. \quad (1)$$

This property was already studied by Trillas-Valverde in 1981 ([19]) and Fodor in 1995 ([10]) for the cases when the negation N is strong, and it was subsequently studied by Jenei in 2000 ([12]) and Balasubramaniam in 2006 ([5]). The equation (1) comes from the classical law of contraposition and it plays an important role in applications like approximate reasoning, deductive systems and formal methods of proof. For more details see Section 1.5 in [4], where more general cases with not necessarily strong negations are dealt with.

However, in many cases the law of contraposition does not hold and thus different ways of modifying an implication have appeared with the aim of obtaining a new implication satisfying the contraposition. These procedures are known as *methods of contrapositivisation* and some of them have appeared, defined with respect to strong negations. For instance, we have the *upper* and *lower* contrapositivisations, introduced in [6] and studied in detail in [10]. Another example is the *medium* contrapositivisation, introduced in [5] (see also Section 7.1 of the book [4]).

It is worth to point up that the study of the contrapositivisation of the residuated implications made in [10] gave rise to the t-norm *Nilpotent minimum*, the first known left-continuous (but not continuous) t-norm, which produced the so important and prolific study of the left-continuous t-norms (see, for instance, [7], [12], [13] and the references therein).

In this paper we present a new method of contrapositivisation with respect to a strong negation N that exhibits very good properties. In particular, our method preserves the usual properties (also preserved by the already existing methods), like the ordering property and the identity principle, but it also preserves the left neutrality principle and other additional properties.

2. Preliminaries

In this section we give some basic definitions and results that will be used along the paper. More details and examples on fuzzy implications can be found in [4], and on negations in [14].

Definition 1 *([9]) A function $N : [0,1] \to [0,1]$ is said to be a* fuzzy negation *if it is decreasing with $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is said to be*

- strict *when it is strictly decreasing and continuous.*
- strong *when it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.*

Definition 2 *([9], [4]) A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an implication function, or an implication, if it satisfies:*

I1) I is decreasing in the first variable and increasing in the second one, that is, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$,

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1, y) \geq I(x_2, y)$$

and

$$\text{if } y_1 \leq y_2, \text{ then } I(x, y_1) \leq I(x, y_2)$$

I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

Among many other properties usually required for fuzzy implications we recall here some of the most important ones.

- $CP(N)$ Law of contraposition with respect to a fuzzy negation N:

$$I(x,y) = I(N(y), N(x)) \text{ for all } x, y \in [0,1].$$

- (EP) Exchange Principle:

$$I(x, I(y,z)) = I(y, I(x,z)) \text{ for all } x, y, z \in [0,1].$$

- (NP) (Left) Neutrality Property:

$$I(1,y) = y \text{ for all } y \in [0,1].$$

- (OP) Ordering Property:

$$I(x,y) = 1 \iff x \leq y \text{ for all } x, y \in [0,1].$$

- (SN) Strong Negation Principle:

$$I(x,0) \text{ is a strong negation for all } x \in [0,1].$$

- (IP) Identity Principle:

$$I(x,x) = 1 \text{ for all } x \in [0,1].$$

The two most usual types of implications are R-implications derived from left-continuous t-norms (and also from more general conjunctive aggregation functions, like uninorms, copulas, quasi-copulas, and so on, see for instance [4, 8, 17]), and (S,N)-implications derived from fuzzy negations and t-conorms (and also from more general disjunctive aggregation functions, see for instance [1, 17]).

Definition 3 ([4]) *Let I be an implication and N a fuzzy negation. We define the $N-$ reciprocal of I as the implication I_N*

$$I_N(x,y) = I(N(y), N(x)) \ , \ \ x, y \in [0,1]$$

Definition 4 ([4]) *Let I be a fuzzy implication. The fuzzy negation N_I defined by*

$$N_I(x) = I(x,0) \text{ for all } x \in [0,1]$$

is called the natural negation of I.

We recall some contrapositivisation techniques proposed in [4].

Definition 5 *Let I be an implication and N a fuzzy negation. The functions I_N^u, I_N^l, $I_N^m : [0,1] \times [0,1] \to [0,1]$ are defined as:*

- Upper contrapositivisation of I with respect to N:

$$I_N^u(x,y) = \max(I(x,y), I_N(x,y)), \quad x, y \in [0,1].$$

- Lower contrapositivisation of I with respect to N:

$$I_N^l(x,y) = \min(I(x,y), I_N(x,y)), \quad x, y \in [0,1].$$

- Medium contrapositivisation of I with respect to N:

$$I_N^m(x,y) = \min(I(x,y) \vee N(x), I(N(y), N(x)) \vee y)$$

for all $x, y \in [0,1]$, where $\vee$ indicates the maximum.

Proposition 1 *Let I be an implication and N a fuzzy negation. The functions I_N^u, I_N^l and I_N^m are fuzzy implications. Moreover, if N is strong, then all of them satisfy $CP(N)$.*

Proposition 2 *Let I be an implication and N a fuzzy negation. If I satisfies (OP) and (IP), then so do I_N^u, I_N^l and I_N^m.*

3. N-lower-contrapositivisation with respect to strong negations

In this section we want to present a new type of contrapositivisation with respect to a strong negation N that allows to transform any implication (not satisfying CP(N)) into another one that satisfies it. We begin giving the general definition with respect to any negation.

Definition 6 *Given an implication I and a negation N, we define the N-lower-contrapositivisation of I, denoted by I_N^{lc}, as the binary operator on* $[0,1]$ *given by*

$$I_N^{lc}(x,y) = \begin{cases} I(x,y) & if\ y \geq N(x) \\ I(N(y),N(x)) & if\ y < N(x) \end{cases} \quad (2)$$

Figure 1 shows the structure of the N-lower-contrapositivisation of an implication I with respect to any negation (for simplicity, the figure represents a strong negation).

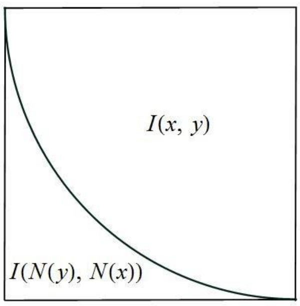

Figure 1: Structure of the N-lower-contrapositivisation of an implication I.

The idea of this definition arose from previous works of the authors. Given any strong negation N, the implication

$$I(x,y) = \begin{cases} 1 & if\ x \leq y \\ 1-x+y & if\ x > y \geq N(x) \\ 1-N(y)+N(x) & if\ x, N(x) > y \end{cases} \quad (3)$$

appears in [1] as the (S,N)-implication obtained from N and the aggregation function $G(x,y) = 1-\max(0, N(x\wedge y) - x \vee y)$, where $\wedge$ indicates the minimum and $\vee$ the maximum. Observe that the above implication is in fact given by the Łukasiewicz implication

$$I_Ł(x,y) = \min(1, 1-x+y), \quad x,y \in [0,1]$$

in the region where $y \geq N(x)$ and it is given by the N-reciprocal of the Łukasiewicz implication in the region where $y < N(x)$.

Another example that appears in [1] is the (S,N)-implication obtained from the strong negation N and the aggregation function

$$G(x,y) = \min\left(1, \frac{x\vee y}{N(x\wedge y)}\right),$$

which is given by

$$I(x,y) = \begin{cases} 1 & if\ x \leq y \\ \frac{y}{x} & if\ x > y \geq N(x) \\ \frac{N(x)}{N(y)} & if\ x, N(x) > y. \end{cases} \quad (4)$$

Similarly as above, it is easy to see that this implication coincides with the Goguen implication

$$I_{GG}(x,y) = \min(1, \frac{y}{x})$$

in the region where $y \geq N(x)$ and it is given by the N-reciprocal of the Goguen implication in the region where $y > N(x)$.

Thus, generalizing this procedure to any implication function I we obtain the definition given in equation (2), and then the implications described above are two direct examples of N-lower-contrapositivisations.

Example 1 *i) Let us consider the Łukasiewicz implication $I = I_Ł$ and a strong negation N. Then $I_N^{lc}(x,y)$ is the implication considered in the equation (3).*

ii) Let us consider now the Goguen's implication $I = I_{GG}$ and a strong negation N. Then $I_N^{lc}(x,y)$ is the implication considered in the equation (4).

All the above examples are related to strong negations, but this condition is not necessary to obtain implications through Definition 6. In what follows we give some examples proving this fact and also that not any negation can be used.

Example 2

1) Let I be an implication and N_0 the smallest negation:

$$N_0(x) = \begin{cases} 0 & if\ x > 0 \\ 1 & if\ x = 0 \end{cases}$$

Then

$$I_{N_0}^{lc}(x,y) = \begin{cases} I(x,y) & if\ x > 0 \\ 1 & if\ x = 0 \end{cases}$$

Therefore $I_{N_0}^{lc} = I$, and thus it is an implication.

2) Let I be an implication and N_1 the greatest negation:

$$N_1(x) = \begin{cases} 1 & if\ x < 1 \\ 0 & if\ x = 1 \end{cases}$$

Then

$$I_{N_1}^{lc}(x,y) = \begin{cases} 1 & if\ x < 1 \\ I(1,y) & if\ x = 1, \end{cases}$$

Therefore $I_{N_1}^{lc}$ is always an implication.

Example 3 *Let us consider the Reichenbach implication,* $I_{RC}(x,y) = 1 - x + xy$, *and the following negation*

$$N(x) = \begin{cases} 1 & if\ x = 0 \\ a & if\ 0 < x < 1 \\ 0 & if\ x = 1 \end{cases}$$

where $a \in (0,1)$.

Let $x, y \in [0,1]$ *such that* $y < a < x$. *Then* $(I_{RC})^{lc}_N(x,y) = I_{RC}(N(y), N(x)) = I_{RC}(a,a) = 1 - a + a^2$, *Whereas* $(I_{RC})^{lc}_N(x,a) = I_{RC}(x,a) = 1 - x + xa < 1 - a + a^2$.

Therefore, $(I_{RC})^{lc}_N$ *is not an implication. The previous negation* N *and the structure of the corresponding* N*-lower-contrapositivisation are shown in Figure 2.*

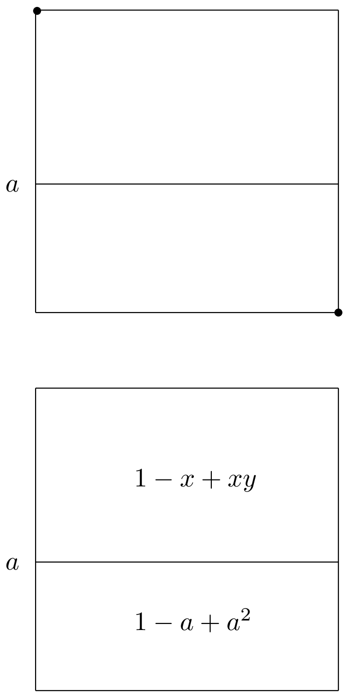

Figure 2: The graph of the negation N in Example 3 (up) and the structure of the N-lower-contrapositivisation of the Reichenbach implication with respect to N (down).

Thus we see that not any negation can be used if we want to obtain implications. Let us see that, in the case of strong negations N, the N-lower-contrapositivisation of any implication I is always an implication.

Proposition 3 *Let* N *be a strong negation and* I *an implication. Then* I^{lc}_N *is an implication.*

Proof. Note that I^{lc}_N clearly satisfies the boundary conditions given by condition $I2$). Thus, to prove that I^{lc}_N is an implication, it is only necessary to show the monotonicities of the horizontal and vertical sections when crossing the graph of the negation N.

- Let $x \in (0,1)$ and $x' < x$. We want to prove that $I^{lc}_N(x', N(x)) \geq I^{lc}_N(x, N(x))$. We have:

  $$I^{lc}_N(x', N(x)) = I(N^2(x), N(x')) = I(x, N(x'))$$

  and, on the other hand, $I^{lc}_N(x, N(x)) = I(x, N(x))$. Then, since $x' < x$, $N(x') \geq N(x)$, and the result follows.
- Now, let us take $x \in (0,1)$ and consider $y' < N(x)$. We want to prove that $I^{lc}_N(x, y') \leq I^{lc}_N(x, N(x))$. But $I^{lc}_N(x, N(x)) = I(x, N(x))$ and, on the other hand, $I^{lc}_N(x, y') = I(N(y'), N(x))$. Then, since $y' < N(x)$, $N(y') \geq N^2(x) = x$, and again the result follows. ■

Next we give a family of negations which are not strong (not even strict) such that the N-lower-contrapositivisation of any implication I is also an implication.

Proposition 4 *Given* $a \in (0,1)$, *let* N_a *be the negation given by*

$$N_a(x) = \begin{cases} 1 & if\ x = 0 \\ N'_a(x) & if\ 0 < x \leq a \\ 0 & if\ a < x \leq 1, \end{cases}$$

where N'_a *is any strong negation on the interval* $(0,a)$. *Then the* N_a*-lower-contrapositivisation* $I^{lc}_{N_a}$ *of any implication* I *is an implication.*

Observe that, although I^{lc}_N is an implication, it does not need to satisfy the contraposition property with respect to N. In fact, this is the case for all the contrapositivisations introduced in the previous proposition. Next example considers the case when $N'_a(x) = a - x$.

Example 4 *Let* I *be an implication and* N_a *the following negation:*

$$N_a(x) = \begin{cases} 1 & if\ x = 0 \\ a - x & if\ 0 < x \leq a \\ 0 & if\ a < x \leq 1. \end{cases}$$

Proposition 4 proves that $I^{lc}_{N_a}$ *is an implication. Nevertheless, in general it does not satisfy* $CP(N)$ *(it is sufficient to consider implications* I *such that* $I(x,y) \neq 1$ *for* $x, y > a$*).*

The structure of the N_a-lower-contrapositivisation of an implication I with respect to the negation N_a given in the previous example can be seen in Figure 3.

Remark 1 *Similar negations to the ones given in the previous example were studied in [7] and they were used to construct left continuous t-norms.*

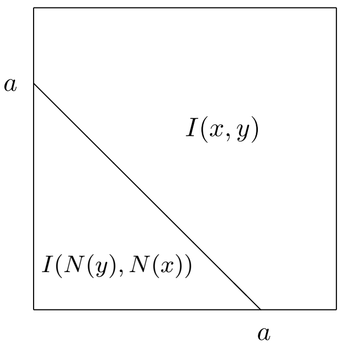

Figure 3: The N_a-lower-contrapositivisation of an implication I.

Next proposition shows that I_N^{lc} satisfies $CP(N)$ when we deal with strong negation.

Proposition 5 *Let I be an implication and N a strong negation. Then I_N^{lc} satisfies $CP(N)$.*

Proof. To prove $CP(N)$, let us consider different cases:

a) If $y = N(x)$ then $N(y) = N^2(x) = x$, and so

$$\begin{aligned} I_N^c(x,y) &= I(x,y) = I(x,N(x)) \\ &= I(N(y),N(x)) \\ &= I_N^c(N(y),N(x)) \end{aligned}$$

b) If $y > N(x)$ then $N(x) < N^2(y) = y$, and so

$$\begin{aligned} I_N^c(x,y) &= I(x,y) = I(N(N(x)),N(N(y))) \\ &= I_N^c(N(y),N(x)). \end{aligned}$$

c) If $y < N(x)$ then $N(x) > N^2(y) = y$, and so

$$I_N^c(x,y) = I(N(y),N(x)) = I_N^c(N(y),N(x)).$$

∎

In fact, only strong negations satisfy this property as it is proved in the following theorem.

Theorem 1 *Let N be a negation. The N-lower-contrapositivisation of any implication I, I_N^{lc}, is an implication that satisfies $CP(N)$ if and only if N is strong.*

Proof. If N is strong we have already proved that I_N^{lc} is an implication function satisfying $CP(N)$ for all implication I.

With respect to the converse, let us give a sketch of the proof. If I_N^{lc} is an implication function satisfying $CP(N)$ for all implication I, then this is true in particular for the following implication:

$$I(x,y) = \begin{cases} y & if\ x = 1 \\ N(x) & if\ y = 0 \\ 1 & otherwise \end{cases}$$

By imposing this fact, one easily concludes that N must be strong. ∎

The rest of the section is devoted to the study of the properties of the N-lower-contrapositivisation when N is a strong negation. First of all, we have that if an implication I already satisfies $CP(N)$, then $I_N^{lc} = I$.

On the other hand, we can see that there exists a connection between I_N^{lc} and the upper contrapositivisation I_N^u (see Preliminaries). Specifically, there are cases in which both contrapositivisations coincide, as the following example shows.

Example 5 *If we consider the Gödel implication*

$$I_{\mathbf{GD}}(x,y) = \begin{cases} 1 & if\ x \le y \\ y & if\ x > y \end{cases}$$

and any strong negation N, then $I_N^{lc} = I_N^u$. Moreover, this implication is given by

$$I_N^{lc}(x,y) = I_N^u(x,y) = \begin{cases} 1 & if\ x \le y \\ \max(N(x),y) & if\ x > y \end{cases}$$

Remark 2 *Observe that the N-lower-contrapositivation (as well as the upper contrapositivation) of the Gödel implication with respect to any strong negation N coincides with the Fodor implication, that is, the residual implication obtained from the minimum nilpotent t-norm. Figure 4 shows the structure of this implication.*

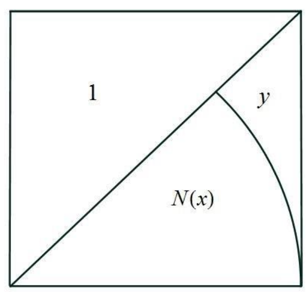

Figure 4: The N-lower-contrapositivisation of the Gödel implication.

Next we see that the ordering property (OP) and the identity principle (IP) are preserved by the N-lower-contrapositivisation, as it is the case of the upper, lower, and medium contrapositivisations.

Proposition 6 *Let I be an implication and N a strong negation.*

1) *If I satisfies the ordering property (OP), then I_N^{lc} also satisfies it.*
2) *If I satisfies (IP), then I_N^{lc} also satisfies it.*

Now we see that the behaviour of I_N^{lc} is even better since it also preserves other additional properties. Let us begin with the (NP) property, which is not preserved in general by the already known contrapositivisations.

Proposition 7 *Let N be a strong negation and I an implication satisfying (NP). Then I_N^{lc} also satisfies (NP).*

We can give the following facts with respect to the natural negation associated to an implication I.

Remark 3 *Let N be a strong negation and I an implication. Then the negation induced by the N-lower-contrapositivisation of I is given by*

$$N_{I_N^{lc}}(x) = I(1, N(x)) \quad \text{for all} \quad x \in [0, 1].$$

Definition 7 *(Definition 7.1.9 in [4]) Let N be a strong negation and I an implication function. Then I_N^{lc} is said to be N-compatible when $N_{I_N^{lc}} = N$.*

Proposition 8 *Let N be a strong negation and I an implication function. Then I_N^{lc} is N-compatible if and only if I satisfies (NP). Moreover, in this case I_N^{lc} also satisfies (SN).*

It can be seen that, in general, I_N^{lc} may not satisfy (SN), even when I satisfies it and N is a strong negation (see the example below). However, in view of Proposition 8, if I satisfies (NP) then I_N^{lc} always satisfies (SN) even if I does not satisfy it.

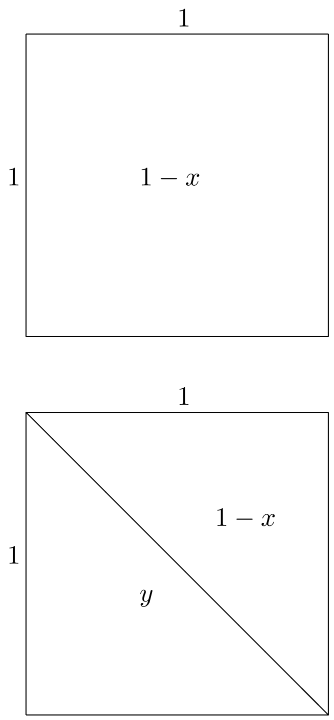

Figure 5: Structure of the implication I (up) and its N-lower-contrapositivisation (down) given in example 6.

Example 6 *Let I be the implication given by*

$$I(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1 \\ 1 - x & \text{otherwise.} \end{cases}$$

Its associated negation is the classical one $N_c(x) = 1 - x$ and so I satisfies (SN). Now let us consider I_N^{lc} the N-lower-contrapositivisation of I with respect to the negation N_c. A simple calculation shows that

$$I_N^{lc}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1 \\ 1 - x & \text{if } 1 - x \leq y < 1 \\ y & \text{if } y < 1 - x < 1. \end{cases}$$

Clearly, the negation associated to I_N^{lc} is the weakest negation and consequently I_N^{lc} does not satisfy (SN).

The structure of I and its N-lower-contrapositivisation I_N^{lc} in this example can be viewed in Figure 5.

On the other hand, the N-lower-contrapositivisation does not preserve the exchange principle (EP), as it is the case of the upper, lower, and medium contrapositivisations. The following example shows that the property (EP) is not generally preserved by I_N^{lc}.

Example 7 *Let us consider the Reichenbach implication, $I_{RC}(x, y) = 1 - x + xy$, which we know that satisfies (EP). Let $N(x) = \sqrt{1 - x^2}$. In this case, the N-lower-contrapositivisation of I is given by*

$$(I_{RC})_N^{lc}(x, y) =$$

$$\begin{cases} 1 - x + xy & if\ y \geq \sqrt{1 - x^2} \\ 1 - \sqrt{1 - y^2}(1 - \sqrt{1 - x^2}) & if\ y < \sqrt{1 - x^2} \end{cases}$$

If we take $x = 1/4, y = 1/2$ and $z = 3/4$, a straightforward calculation proves that $(I_{RC})_N^{lc}(1/4, (I_{RC})_N^{lc}(1/2, 3/4)) = 0.9869$, whereas $(I_{RC})_N^{lc}(1/2, (I_{RC})_N^{lc}(1/4, 3/4)) = 0.9895$. Thus $(I_{RC})_N^{lc}$ does not satisfy (EP).

4. A small variant contrapositivisation

We have presented in previous sections the N-lower-contrapositivisation of implication functions with respect to a strong negation N (see Definition 6), along with some of their properties. Note however that this definition can be slightly modified obtaining a new contrapositivisation technique. The idea is simply to use the N-reciprocal in the region over the negation rather than in the region below the negation. Specifically, we can give the following definition.

Definition 8 *Given an implication I and a negation N, we define the N-upper-contrapositivisation of I, denoted by I_N^{uc}, as the binary operator on $[0, 1]$ given by*

$$I_N^{uc}(x, y) = \begin{cases} I(N(y), N(x)) & if\ y > N(x) \\ I(x, y) & if\ y \leq N(x) \end{cases} \quad (5)$$

Figure 6 shows the structure of the N-upper-contrapositivisation of an implication I with respect to any negation (for simplicity, the figure represents a strong negation).

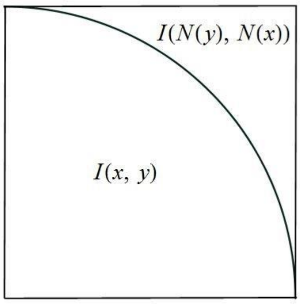

Figure 6: Structure of the N-upper-contrapositivisation of an implication I.

The major part of the results given in the section above for the N-lower-contrapositivisation can be also proved for the N-upper-contrapositivisation. First of all, we have the following result.

Proposition 9 *Let N be a strong negation and I an implication. Then I_N^{uc} is always an implication function that satisfies $CP(N)$*

Again this is true only for strong negations.

Theorem 2 *Let N be a negation. The N-upper-contrapositivisation of any implication I, I_N^{uc}, is an implication that satisfies $CP(N)$ if and only if N is strong.*

The following example shows that in some cases this new contrapositivisation method coincides with the lower-contrapositivisation (see Definition 5 in the Preliminaries).

Example 8 *Let us consider the Gödel implication I. Then, $I_N^{uc} = I_N^l$ for any strong negation N and this implication is given by*

$$I_N^{uc}(x,y) = I_N^l(x,y) = \begin{cases} 1 & if\ x \leq y \\ \min(N(x), y) & if\ x > y \end{cases}$$

Moreover, it can also be proved that this contrapositivisation technique also preserves (OP) and (IP) but not (EP).

However, the behaviour of I_N^{uc} is different from the behaviour of I_N^{lc} with respect to the properties (NP), (SN), and N-compatibility as we can see in the following propositions.

Proposition 10 *Let N be a strong negation and I an implication. The following items hold.*

1) *The natural negation of I_N^{uc} coincides with N_I. That is, $N_{I_N^{uc}} = N_I$.*
2) *I_N^{uc} satisfies (SN) if and only if I satisfies it.*
3) *I_N^{uc} is N-compatible if and only if I is N-compatible. Moreover, in this case I_N^{uc} satisfies (SN).*

Proposition 11 *Let N be a strong negation and I an implication. If I is N-compatible, then I_N^{uc} satisfies (NP).*

Remark 4 *Note that, depending on the strong negation N we deal with, we can choose a contrapositivisation or the other one, in order to modify the given implication I as little as possible. Specifically, if N is given by a convex strong negation near to the smallest negation, the N-lower-contrapositivisation could be used because the region where the initial I is modified (that is, the region under the negation N) is smaller (see Figure 1). Whereas if N is given by a concave strong negation near to the greatest negation, the N-upper-contrapositivisation would be preferred by the same reason (see Figure 6).*

5. Conclusions

One of the most usual and required properties of a fuzzy implication function is the so-called *law of contraposition* or *contrapositive symmetry* with respect to a strong negation N, $CP(N)$. Such a property is important in many application fields like approximate reasoning, deductive systems, decision support systems, formal methods of proof, and for this reason it has been extensively studied in the literature. Unfortunately, there are many implication functions that do not satisfy the law of contraposition and this fact has led to study some techniques of *contrapositivisation*, that is, some methods of modifying a given implication function I that does not satisfy $CP(N)$ in order to obtain a new implication satisfying it.

In this work we have introduced two new contrapositivisation techniques (called N-lower-contrapositivisation and N-upper-contrapositivisation) and we have studied the properties that such methods preserve. We have seen that they retain all properties preserved by the already known methods and also some additional ones. An additional advantage of these new methods lies in the fact that the region where the implication I is modified does not depend on the proper implication I, but only on the considered negation N. Moreover, this region consists of the set of points (x, y) that are under the graph of the negation N (for the N-lower-contrapositivisation) or over that graph (for the N-upper-contrapositivisation).

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