A generalized α-level decomposition concept for numerical fuzzy calculus

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Abstract

This paper presents a new concept for the decomposition of fuzzy numbers into a finite number of α-cuts. Instead of subdividing the μ axis in an equidistant way, we suggest to subdivide the x axis equidistantly leading to a more efficient decomposition of the μ axis. Considering the interpolation error as a measure for the loss of information during the decomposition, our concept leads to the minimal information loss of the decomposed fuzzy numbers.

Keywords: Decomposition of fuzzy numbers, α-cuts, numerical fuzzy calculus

1. Introduction

The question of this paper concerns how to decompose fuzzy numbers into a finite number of α-cuts in the most efficient way. Common approaches to numerical fuzzy calculus [1] use an equidistant subdivision of the μ axis into m intervals of equal length. Unfortunately, for fuzzy numbers without compact support, this procedure has some disadvantages [2]: (1) A 0-cut has to be specially defined. (2) The decomposition number is limited. Furthermore, during the decomposition, no information about the derivatives of the membership functions at the decomposition points is stored. For this reason, the fuzzy numbers can only be reconstructed by means of linear spline interpolation. In case of triangular fuzzy numbers, no information is lost due to constant slopes, and after the decomposition, the fuzzy numbers can be completely reconstructed into the original ones. In case of nonlinear fuzzy numbers, however, a loss of information occurs, which is tending to zero for $m \to \infty$. The question that now arises is how this loss of information can be minimized at a finite decomposition number m.

2. Fuzzy numbers

Fuzzy numbers are a special class of fuzzy sets [3], which can be defined as follows [4]:

A normal, convex fuzzy set $\tilde{x}$ over the real line $\mathbb{R}$ is called *fuzzy number* if there is exactly one $\bar{x} \in \mathbb{R}$ with $\mu_{\tilde{x}}(\bar{x}) = 1$ and the membership function is at least piecewise continuous. The value $\bar{x}$ is called the *modal* or *peak value* of $\tilde{x}$.

2.1. Parametric representation

For an efficient representation of fuzzy numbers, it is useful to introduce a parametric notation. One way to do so is to use two reference functions, which can be defined as follows [4]:

A function $S\colon [0, \infty) \to [0, 1]$ with $S = S(u)$ and $u = u(x)$ is called *reference* (or *shape*) *function* if it satisfies the following conditions:

1. $S(0) = 1$,
2. S is monotonically decreasing in $[0, \infty)$, and
3. $S(1) = 0$ or $\lim_{u\to\infty} S(u) = 0$.

A fuzzy number $\tilde{x}$ is called *LR fuzzy number* if two shape functions L (for the left branch) and R (for the right branch) as well as two parameters $\delta^{\mathrm{L}}, \delta^{\mathrm{R}} \in \mathbb{R}^+$ exist such that

$$\mu_{\tilde{x}}(x) = \begin{cases} L\left(\dfrac{\bar{x} - x}{\delta^{\mathrm{L}}}\right), & x \le \bar{x}, \qquad (1a) \\[2ex] R\left(\dfrac{x - \bar{x}}{\delta^{\mathrm{R}}}\right), & x > \bar{x}, \qquad (1b) \end{cases}$$

where $\bar{x}$ denotes the *modal value*, δ^{L} the *left-hand*, and δ^{R} the *right-hand deviation* of $\tilde{x}$.

In order to decompose LR fuzzy numbers into α-cuts, they must have a compact support. This is the case if $S(1) = 0$; otherwise, the membership function $\mu_{\tilde{x}}(x)$ has to be truncated at $\bar{x} - r\delta^{\mathrm{L}}$ and $\bar{x} + r\delta^{\mathrm{R}}$, respectively, with $r \in \mathbb{R}^+$. The *0-cut* of $\tilde{x}$ is then defined by

$$x(0) = [\bar{x} - r\delta^{\mathrm{L}}, \bar{x} + r\delta^{\mathrm{R}}].$$

2.2. Types of fuzzy numbers

Theoretically, an infinite number of possible types of fuzzy numbers can be defined. However, only few of them are important for engineering applications [2]. These typical fuzzy numbers shall be described in the following.

2.2.1. Triangular fuzzy numbers

Due to its very simple, linear membership function, the *triangular fuzzy number* (TFN) is the most frequently used fuzzy number in engineering. In order to define a TFN with the shape functions

$$L(u) = R(u) = S(u) = 1 - u$$

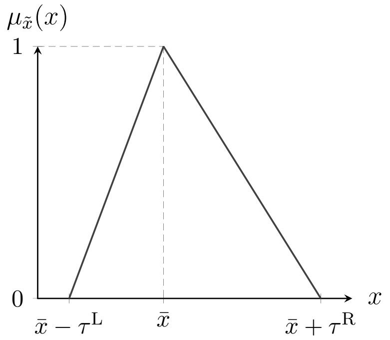

Figure 1: Triangular fuzzy number.

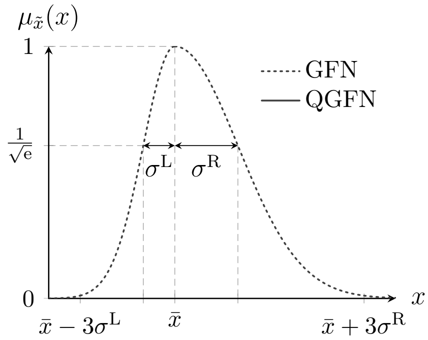

Figure 2: (Quasi-)Gaussian fuzzy number.

and the membership function

$$\mu_{\tilde{x}}(x)=\begin{cases}1+\dfrac{x-\bar{x}}{\tau^{\mathrm{L}}}, & \bar{x}-\tau^{\mathrm{L}}\le x\le\bar{x},\\[2mm] 1-\dfrac{x-\bar{x}}{\tau^{\mathrm{R}}}, & \bar{x}<x\le\bar{x}+\tau^{\mathrm{R}},\end{cases} \tag{2}$$

we use the parametric notation [2]

$$\tilde{x}=\mathrm{tfn}(\bar{x},\tau^{\mathrm{L}},\tau^{\mathrm{R}}),$$

where $\bar{x}$ denotes the *modal value*, τ^{L} the *left-hand*, and τ^{R} the *right-hand spread* of $\tilde{x}$, see Figure 1. If $\tau^{\mathrm{L}}=\tau^{\mathrm{R}}$, the TFN is called *symmetric*. Its α-cuts $x(\alpha)=[x^{\mathrm{L}}(\alpha),x^{\mathrm{R}}(\alpha)]$ result from the inverse functions of Eqs. (2) with respect to x:

$$x^{\mathrm{L}}(\alpha)=\bar{x}-\tau^{\mathrm{L}}(1-\alpha),\quad 0<\alpha\le 1,$$
$$x^{\mathrm{R}}(\alpha)=\bar{x}+\tau^{\mathrm{R}}(1-\alpha),\quad 0<\alpha\le 1.$$

2.2.2. Gaussian fuzzy numbers

Another widely-used fuzzy number in engineering is the *Gaussian fuzzy number* (GFN), which is based on the normal distribution from probability theory. In order to define a GFN with the shape functions

$$L(u)=R(u)=S(u)=\exp(-u^2/2)$$

and the membership function

$$\mu_{\tilde{x}}(x)=\begin{cases}\exp\left[-\dfrac{1}{2}\left(\dfrac{x-\bar{x}}{\sigma^{\mathrm{L}}}\right)^2\right], & x\le\bar{x},\\[2mm] \exp\left[-\dfrac{1}{2}\left(\dfrac{x-\bar{x}}{\sigma^{\mathrm{R}}}\right)^2\right], & x>\bar{x},\end{cases}$$

we use the parametric notation [2]

$$\tilde{x}=\mathrm{gfn}(\bar{x},\sigma^{\mathrm{L}},\sigma^{\mathrm{R}}),$$

where $\bar{x}$ denotes the *modal value*, σ^{L} the *left-hand*, and σ^{R} the *right-hand standard deviation* of $\tilde{x}$, see Figure 2. If $\sigma^{\mathrm{L}}=\sigma^{\mathrm{R}}$, the GFN is called *symmetric*. Its α-cuts $x(\alpha)=[x^{\mathrm{L}}(\alpha),x^{\mathrm{R}}(\alpha)]$ result in

$$x^{\mathrm{L}}(\alpha)=\bar{x}-\sigma^{\mathrm{L}}\sqrt{-2\ln(\alpha)},\quad 0<\alpha\le 1,$$
$$x^{\mathrm{R}}(\alpha)=\bar{x}+\sigma^{\mathrm{R}}\sqrt{-2\ln(\alpha)},\quad 0<\alpha\le 1.$$

2.2.3. Quasi-Gaussian fuzzy numbers

Since GFNs do not have a compact support, it is useful to truncate their membership function at $\bar{x}-3\sigma^{\mathrm{L}}$ and $\bar{x}+3\sigma^{\mathrm{R}}$, respectively. In order to define a *quasi-Gaussian fuzzy number* (QGFN) [2] with the shape functions

$$L(u)=R(u)=S(u)=\exp(-u^2/2)$$

and the membership function

$$\mu_{\tilde{x}}(x)=\begin{cases}\exp\left[-\dfrac{1}{2}\left(\dfrac{x-\bar{x}}{\sigma^{\mathrm{L}}}\right)^2\right], & \bar{x}-3\sigma^{\mathrm{L}}\le x\le\bar{x},\\[2mm] \exp\left[-\dfrac{1}{2}\left(\dfrac{x-\bar{x}}{\sigma^{\mathrm{R}}}\right)^2\right], & \bar{x}<x\le\bar{x}+3\sigma^{\mathrm{R}},\end{cases}$$

we use the parametric notation

$$\tilde{x}=\mathrm{qgfn}(\bar{x},\sigma^{\mathrm{L}},\sigma^{\mathrm{R}}),$$

where again $\bar{x}$ denotes the *modal value*, σ^{L} the *left-hand*, and σ^{R} the *right-hand standard deviation* of $\tilde{x}$, see Figure 2. If $\sigma^{\mathrm{L}}=\sigma^{\mathrm{R}}$, the QGFN is called *symmetric*. Its α-cuts $x(\alpha)=[x^{\mathrm{L}}(\alpha),x^{\mathrm{R}}(\alpha)]$ are

$$x^{\mathrm{L}}(\alpha)=\bar{x}-\sigma^{\mathrm{L}}\sqrt{-2\ln(\alpha)},\quad \exp(-9/2)\le\alpha\le 1,$$
$$x^{\mathrm{R}}(\alpha)=\bar{x}+\sigma^{\mathrm{R}}\sqrt{-2\ln(\alpha)},\quad \exp(-9/2)\le\alpha\le 1.$$

2.3. Decomposition of fuzzy numbers

According to the *decomposition theorem* [5], every fuzzy set $\tilde{A}$ can be uniquely represented by the union of its α-cuts:

$$\tilde{A}=\bigcup_{\alpha}\alpha A(\alpha). \tag{3}$$

For practical applications, however, the infinite number of α-cuts in Eq. (3) has to be reduced to a finite number. This is usually done by subdividing the interval $[0,1]$ of the μ axis into m intervals of equal length. The discrete values α_j of the $(m+1)$ levels of membership are then given by [2]

$$\alpha_j=\frac{j}{m},\quad j=0,\ldots,m. \tag{4}$$

The parameter m in Eq. (4) is usually referred to as the *decomposition number*.

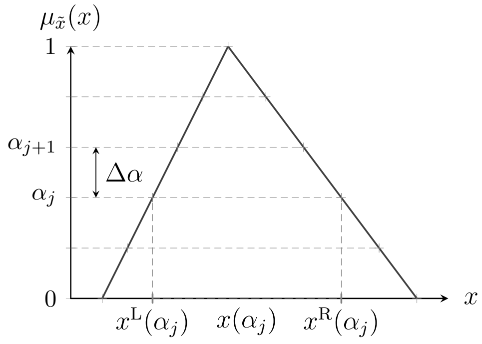

Figure 3: Decomposition of a fuzzy number $\tilde{x}$ into α-cuts.

Applying the decomposition theorem to a finite number of α-cuts, a fuzzy number $\tilde{x}$ can be expressed by the following set of $(m+1)$ intervals:

$$X = \{x(\alpha_0), \ldots, x(\alpha_m)\}$$

with $x(\alpha_j) = [x^{\mathrm{L}}(\alpha_j), x^{\mathrm{R}}(\alpha_j)]$, $j = 0, \ldots, m$, and $x(\alpha_0) = \mathrm{supp}(\tilde{x}) \cup \{\bar{x} - r\delta^{\mathrm{L}}, \bar{x} + r\delta^{\mathrm{R}}\}$, see Figure 3.

3. New decomposition concept

Let

$$\Delta = \{x_0, \ldots, x_n \,|\, \bar{x} - r\delta^{\mathrm{L}} = x_0 < \cdots < x_n = \bar{x} + r\delta^{\mathrm{R}}\}$$

be a partition of the interval $I = [\bar{x} - r\delta^{\mathrm{L}}, \bar{x} + r\delta^{\mathrm{R}}]$ and

$$S_\Delta = \{s \in C_I \,|\, s_{[x_{i-1}, x_i]} \in \Pi_1, i = 1, \ldots, n\}$$

the set of linear splines on I to this partition, where C_I denotes the set of continuous functions on I, $s_{[x_{i-1},x_i]}$ the linear spline on $[x_{i-1}, x_i]$, and Π_1 the set of polynomials with degree one. Then, the following error estimate holds [6]:

$$\|(\mu_{\tilde{x}} - s)(x)\|_\infty \le \frac{h^2}{8} \|\mu''_{\tilde{x}}(x)\|_\infty$$

with

$$h = \max_{i=1,\ldots,n} \{x_i - x_{i-1}\}. \tag{5}$$

In this context, we define the interpolation error e as

$$e := \frac{\|(\mu_{\tilde{x}} - s)(x)\|_\infty}{\|\mu''_{\tilde{x}}(x)\|_\infty} \le \frac{h^2}{8}.$$

According to Eq. (5), the interpolation error is minimal if the partition Δ is equidistant. Considering the interpolation error as a measure for the loss of information during the decomposition, an equidistant partition leads to the minimal information loss of the decomposed fuzzy numbers. Therefore, we suggest to decompose the x axis equidistantly with

$$\Delta x^{\mathrm{L}} = h_i = x_i - x_{i-1} = \frac{r\delta^{\mathrm{L}}}{m}, \quad i = 1, \ldots, n,$$

for the left and

$$\Delta x^{\mathrm{R}} = h_i = x_i - x_{i-1} = \frac{r\delta^{\mathrm{R}}}{m}, \quad i = 1, \ldots, n,$$

for the right branch of an LR fuzzy number.

Consequently, substituting

$$x = \bar{x} - r\delta^{\mathrm{L}} + j\frac{r\delta^{\mathrm{L}}}{m}, \quad j = 0, \ldots, m,$$

into Eq. (1a) leads to

$$\alpha_j^{\mathrm{L}} = L\Big[r\Big(1 - \frac{j}{m}\Big)\Big], \quad j = 0, \ldots, m,$$

and

$$x = \bar{x} + r\delta^{\mathrm{R}} - j\frac{r\delta^{\mathrm{R}}}{m}, \quad j = 0, \ldots, m,$$

into Eq. (1b) to

$$\alpha_j^{\mathrm{R}} = R\Big[r\Big(1 - \frac{j}{m}\Big)\Big], \quad j = 0, \ldots, m,$$

which is independent of $\bar{x}$, δ^{L}, and δ^{R}!

Since for the typical fuzzy numbers from Section 2.2,

$$L(u) = R(u) = S(u),$$

we suggest to decompose these fuzzy numbers at

$$\alpha_j = S\Big[r\Big(1 - \frac{j}{m}\Big)\Big], \quad j = 0, \ldots, m. \tag{6}$$

In particular, for TFNs with $r = 1$, we get

$$\alpha_j = \frac{j}{m}, \quad j = 0, \ldots, m,$$

and for QGFNs with $r = 3$,

$$\alpha_j = \exp\left[-\frac{9}{2}\Big(1 - \frac{j}{m}\Big)^2\right], \quad j = 0, \ldots, m. \tag{7}$$

Hence, the classical decomposition concept from Eq. (4) is only a special case of the general decomposition scheme according to Eq. (6) for TFNs.

Since for nonlinear fuzzy numbers an equidistant decomposition of the μ axis results in a non-equidistant decomposition of the x axis, the estimate of the interpolation error for a symmetric QGFN according to the classical and the new decomposition concept shall be compared in the following.

For a symmetric QGFN with $\sigma^{\mathrm{L}} = \sigma^{\mathrm{R}} = \sigma$, the estimate of the interpolation error according to the classical decomposition concept can be expressed by

$$\frac{e_{\mathrm{cl}}}{\sigma^2} \le \frac{1}{8}\left(3 - \sqrt{2\ln(m)}\right)^2. \tag{8}$$

On the other hand, the estimate of the interpolation error for the new decomposition scheme is

$$\frac{e_{\mathrm{new}}}{\sigma^2} \le \frac{9}{8}\frac{1}{m^2}. \tag{9}$$

The plots of Eqs. (8) and (9) in the typical range $m \in \{3, \ldots, 10\}$ for practical applications are illustrated in Figure 4. There, we can see that with the new decomposition concept, the interpolation error can be significantly reduced compared to the classical decomposition scheme.

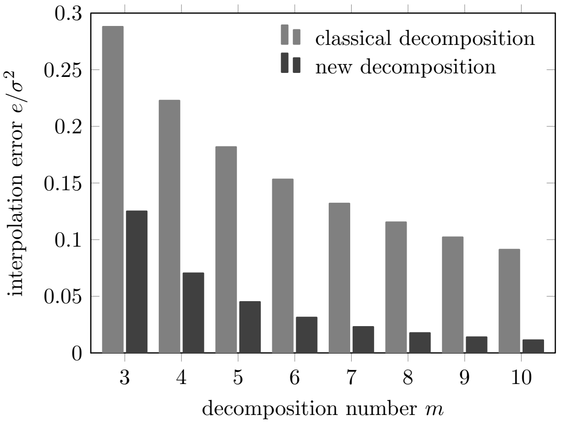

Figure 4: Comparison of the estimate of the interpolation error for a QGFN according to the classical and the new decomposition concept.

4. Engineering application

In order to illustrate the proposed decomposition concept in a more practical context, we consider a rather simple but typical example from engineering mechanics consisting of a two-component massless rod under tensile load [2]. The left component of the rod (stiffness c_1) is clamped to a wall, whereas the right component (stiffness c_2) is subjected to a tensile force F. The (static) displacement u of the tip of the rod is determined by

$$u = \left(\frac{1}{c_1} + \frac{1}{c_2}\right) F.$$

The first component is assumed to be made of steel and the second component of aluminum with the following nominal stiffness values:

$$c_1 = 4.0 \cdot 10^4 \, \mathrm{N/mm}, \\ c_2 = 1.035 \cdot 10^4 \, \mathrm{N/mm}.$$

In reality, however, exact stiffness values for both rod components can usually not be provided due to variations in the manufacturing process. In order to include these uncertainties into the computation, the stiffness parameters c_1 and c_2 shall be modeled as symmetric QGFNs with the nominal values as modal values. The standard deviations of $\tilde{c}_1$ and $\tilde{c}_2$ are assumed to be 5 % of their modal values.

The α-cuts $u(\alpha_j) = [u^{\mathrm{L}}(\alpha_j), u^{\mathrm{R}}(\alpha_j)]$ of $\tilde{u}$ can be computed from [7]

$$u^{\mathrm{L}}(\alpha_j) = u\big(c_1^{\mathrm{R}}(\alpha_j), c_2^{\mathrm{R}}(\alpha_j)\big), \\ u^{\mathrm{R}}(\alpha_j) = u\big(c_1^{\mathrm{L}}(\alpha_j), c_2^{\mathrm{L}}(\alpha_j)\big). \tag{10}$$

The plots of Eqs. (10) for the classical and the new decomposition concept for $m = 4$ as well as the exact solution are illustrated in Figure 5. There, we can see that for medium membership values, both concepts lead to a very good approximation of the exact solution, whereas for high membership values, the approximation is very poor. In the lower range of the membership values, however, the new decomposition concept provides a better approximation of the exact solution compared to the classical one.

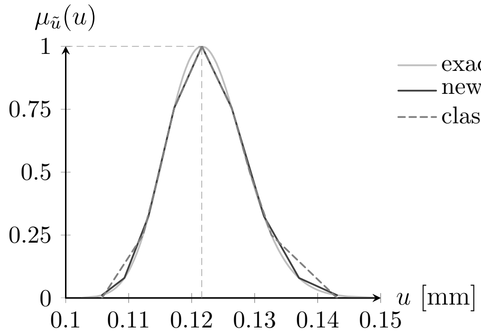

Figure 5: Fuzzy displacement $\tilde{u}$ of the tip of the rod for the classical and the new decomposition concept compared to the exact solution.

5. Conclusions

We introduced a new concept for the decomposition of fuzzy numbers into a finite number of α-cuts. Considering the interpolation error as a measure for the loss of information during the decomposition, our concept leads to the minimal information loss of the decomposed fuzzy numbers. Furthermore, for fuzzy numbers without a compact support, no 0-cut has to be specifically defined, and the decomposition number is no longer limited.

Since in practical applications mostly TFNs and QGFNs are used [2], we suggest to use the decomposition concept according to Eq. (7) instead of the classical decomposition scheme according to Eq. (4).

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