The Binary Minimum-energy Parseval Frames and Pseudo-frames and Applications in Economics and Management

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Abstract. Material science is an interdisciplinary field applying the properties of matter to various areas of science and engineering. Frames have become the focus of active research field, both in the -ory and in applications. In the article, the binary minimum-energy wavelet frames and frame multi -resolution resolution are introduced. A precise existence criterion for minimum-energy frames in terms of an ineqity condition on the Laurent poly-nomial symbols of the filter functions is provided. An explicit formula for designing minimum-energy frames is also established. The sufficient condi tion for the existence of tight wavelet frames is obtained by virtue of a generalized multiresolution analysis.

Introduction

Management Science is a scholarly journal that publishes scientific research into the practice of management. Our scope includes articles that address management issues with tools from foundati -onal fields such as computer science, economics, mathematics, operations research, political scien -ce, psychology, sociology, and statistics, as well as cross-functional, multidisciplinary research that reflects the diversity of the management science professions. Since the late 20th century, multiwavelet theory have become the focus in the wavelet analysis. Multiwavelets, which can offer properties like symmetry, orthogonality, short support at the same time, have been widely used in signal procssing, image processing and so on. The notion of vector-valued wavelets is introduced, the existence and the method for construction of the orthogonal vector-valued wavelets are studied in [1] by Xia and Suter. As reported by Xia, multivelets can be generated from the component func -tions in vector-valued wavelets. Fowler and Li using discrete biorthogonal vector-valued wavelet tr -ansform to study Marine eddy current pheno- menon. To obtain some beautiful features, S.Yang [2] introduce the concepts of two-direction scaling function and two-direction wavelets and give the definition of orthogonal two-direction scaling function and construct the orthogonal two-direction wavelets. Through the rational scrambling on the carrier image and implementing M-band wavelet transform, the carrier image is decomposed into according the low frequency subband images. Secondly, the wavelet subbands are separated into blocks and the singular value decomposition is implemented with each of the blocks to produce sub-blocks. Comparing characteristic value of each sub-blocks and the mean value of each sub-blocks, we can obtain the transition matrix. Then, the matrix is co -mbined with a visual secret sharing algorithm to create the main sharing. Finally, the main sharing is combined with the secret watermarking information to generate the zero-watermark. Experimental results show that the combination of the algorithm with M-band wavelet algorithm not only has higher security and robustness, but also better than the results obtained by the dyadic wavelets. In medical imaging, we can get ecg compression algorithm by using two-direction wavelets transform in [3]. The rise of frame theory in applied mathematics is due to the flexibility and redundancy of frames, where robustness, error tolerance and noise suppression play a vital role [4, 5]. The concept of frame multiresolution anaysis (FMRA) is described in [6] generalizes the notion of MRA by allowing non-exact affine frames.

Binary frame multiresolution analysis

In this paper,we use the following notations. Let Z be the set of integers and R be the aggregate of real numbers. C stands for a set of complex numbers. We consider any functions $\psi(x)$, $\lambda(x)$ of two variables in the space $L^2(R^2)$ with the inner product:

$$\langle \lambda, \psi \rangle = \int_{R^2} \lambda(x)\,\overline{\psi(x)}\,dx, \quad \widehat{\psi}(\omega) = \int_{R^2} \psi(x)\, e^{-x\cdot\omega} dx .$$

As usual, $\hat{\hbar}(\omega)$ denotes the Fourier transform of any function $\hbar(x) \in L^2(R^2)$. Let Ω be a separable Hilbert space and Λ be an index set. We recall that a sequence $\{\Gamma_\iota : \iota \in \Lambda\} \subseteq \Omega$ is a frame for Ω if there exist two positive constants A_1, A_2 such that

$$\forall\ \xi \in \Omega, \qquad A_1 \| \xi \|^2 \le \sum_{\iota\in\Lambda} \left|\langle \xi, \Gamma_\iota \rangle\right|^2 \le A_2 \| \xi \|^2, \tag{1}$$

where $\| \xi \|^2 = \langle \xi, \xi \rangle$, and A_1, A_2 are called frame bounds.

For any $f(t) \in L^2(R)$, the novel fractional Fourier transformis defined as

$$F_\alpha(u) = \mathcal{F}^\alpha\{f(t)\}(u) = \int_{\mathbb{R}} f(t)\mathcal{K}_\alpha(u,t)dt$$

A sequence $\{\hbar_\iota : \iota \in \Lambda\} \subseteq \Omega$ is a tight one if we can choose $A_1 = A_2$. A frame $\{\hbar_\iota : \iota \in \Lambda\}$ is an exact frame if it ceases to be a frame when any one of its elements is removed. If $A_1 = A_2 = 1$, then it follows from (1) that $\forall\ \xi \in \Omega,\ \ \xi = \sum_{\iota\in\Lambda} \langle \xi, \Gamma_\iota \rangle \Gamma_\iota$.

$$f(t) = \frac{1}{2\pi C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} W_f^\alpha(a,b) f(t) \psi_{\alpha,a,b}(t) \frac{da}{a^2} db$$

A sequence $\{\Gamma_\iota : \iota \in \Lambda\} \subseteq \Omega$ is a Bessel sequence if (only) the upper inequality of (1) follows. If only for all $\hbar \in U \subset \Omega$, the upper inequality of (1) holds, the sequence $\{\Gamma_\iota : \iota \in \Lambda\} \subseteq \Omega$ is a Bessel sequence with respect to (w.r.t.) U. Moreover, we assume that $U \subset \Omega$ is a closed subspace, if for all $\lambda \in U$, there exist two real numbers $A_1, A_2 > 0$ such that $A_1 \| \lambda \|^2 \le \sum_{\iota\in\Lambda} |\langle \lambda, \Gamma_\iota \rangle|^2 \le A_2 \| \lambda \|^2$, the sequence $\{\Gamma_\iota : \iota \in \Lambda\} \subseteq \Omega$ is called an U – subspace frame.

Theorem1[4]. Let $\{f_k\}_{k=1}^\infty$ be a frame for a Hilbert space Ω with frame operators S .Then

(1) S is invertible and self-adjoint. (2) Every $f \in \Omega$ can be represented as

$$f = \sum_{k=1}^\infty \langle f, S^{-1} f_k \rangle f_k = \sum_{k=1}^\infty \langle f, f_k \rangle S^{-1} f_k$$

(2) Every $f \in \Omega$ has the representation $f = \sum_{k=1}^\infty c_k f_k$ for some scalar coefficients $\{c_k\}_{k=1}^\infty$, then

$$\sum_{k=1}^\infty | c_k |^2 = \sum_{k=1}^\infty |< f, S^{-1} f_k >|^2 + \sum_{k=1}^\infty | c_k - < f, S^{-1} f_k >|^2$$

If $\{\Gamma_\iota : \iota \in \Lambda\} \subseteq \Omega$ is a frame for Ω, then there exist a dual frame $\{\Gamma_\iota^* : \iota \in \Lambda\} \subseteq \Omega$ such that

$$\forall \xi \in \Omega, \quad \xi = \sum_{\iota\in\Lambda} \langle \xi, \Gamma_\iota \rangle \Gamma_\iota^* = \sum_{\iota\in\Lambda} \langle \xi, \Gamma_\iota^* \rangle \Gamma_\iota . \tag{2}$$

For $c=\{c(v)\}\in \ell^2(Z^2)$, its discrete-time Fourier transform is as the function in $L^2(0,1)^2$ by

$$Fc(\omega)=C(\omega)=\sum_{v\in Z^2} c(v)\,e^{-2\pi i x\omega}dx\,, \tag{3}$$

Note that the discrete-time Fourier transform is 1-periodic. Let $T_v h(x)$ denote integer translates of a function $h(x)\in L^2(R^2)$, i.e., $(T_v h)(x)=h(x-v)$,and let $V_0=\overline{span}\{T_v h: v\in Z^2, h(x)\in L^2(R^2)\}$ be a cl -osed subspace of space $L^2(R^2)$. Assume that $H(\omega):=\sum_v |\hat{h}(\omega+v)|^2 \in L^\infty[0,1]^2$. In $[5]$, the sequen -ce $\{T_v h(x)\}_v$ is a frame for V_0 if and only if there exist positive constants A_1 and A_2 such that

$$A_1\le H(\omega)\le A_2\,,\ \text{a.e.}, \omega\in[0,1]^2\setminus\mathbb{N}=\{\omega\in[0,1]^2: H(\omega)\ne 0\}. \tag{4}$$

Let $\{T_v\hbar(x)\}$ and $\{T_v\tilde{\hbar}(x)\}$ $(v\in Z^2)$ be two sequences in $L^2(R^2)$ and U be a closed subspace of $L^2(R^2)$. We say that $\{T_v\hbar(x)\}$ forms a pseudoframe for the subspace U with respect to (w.r.t.) $\{T_v\tilde{\hbar}(x)\}$ if

$$\forall\, f(x)\in U\,, \qquad f(x)=\sum_{v\in Z^2}\langle f, T_v\hbar\rangle T_v\tilde{\hbar}(x)\,. \tag{5}$$

Definition 1. $(\{V_k\}_{k\in Z},\phi)$ is said to be a frame multiresolution analysis of space $L^2(R^2)$ if each V_k Is a closed subspace of $L^2(R^2)$ and a binary function $\phi\in V_0$ such that ① $\cdots\subset V_{-1}\subset V_0\subset V_1\subset\cdots$ ② $\bigcap_{k\in Z}V_k=\{0\}$; $\overline{\bigcup_{k\in Z}V_k}$ is dense in $L^2(R^2)$; ③ $h(x)\in V_n$ if and only if $\psi(ax)\in V_{n+1}$ $\forall n\in Z$; ④ $\psi(x)\in V_0$ implies $T_u h(x)\in V_0$, for $u\in Z^2$; (v) the family $\{T_u\phi(x): u\in Z^2\}$ is a frame for V_0.

Definition 2. If the condition (v) in the above Definition 1 can be substituted for the condition that $\{T_u\phi(x):\ u\in Z^2\}$ constitutes a pseudoframe for V_0 with respect to $\{T_u\tilde{\phi}(x):\ u\in Z^2\}$, We say that $\{V_n\}_{n\in Z}$ is a bivariate generalized multiresolution structure (BGMS)

Definition 3. Let $\phi(x)\in L^2(R^2)$, with $\hat{\phi}(\omega)\in L^\infty(R^2)$, continuous at 0, and $\hat{\phi}(0)=1$, be a refinable function, which generates the nested subspaces $\{V_n\}$ in the sense of (i). Then a family of finite fu- -nctions $\Psi=\{\psi^1(x),\psi^2(x),\cdots,\psi^r(x)\}$ is called a binary minimum-energy (wavelet) frame associ- -ated with $\phi(x)$, if for arbitrary binary function $\lambda(x)\in L^2(R^2)$, it follows that

$$\sum_{v\in Z^2}|<\lambda,\phi_{1,v}>|^2=\sum_{v\in Z^2}|<\lambda,\phi_{0,v}>|^2+\sum_{\iota=1}^{r}\sum_{v\in Z^2}|<\lambda,\psi^{\iota}_{0,v}>|^2\,, \tag{6}$$

Where $r\in N$ is a constant integer and $\phi_{n,v}(x)=2^n\phi(2^n x-v)$, and $\psi^{\iota}_{n,v}(x)=2^n\psi^{\iota}(2^n x-v), v\in Z^2$.

Definition 4. Let $a>1, b>0$ be two constants and $\psi\in L^2(R^2)$. A frame for the space $L^2(R^2)$ of the form $\{a^{j/2}\psi(a^j x-kb)\}_{j,k\in Z}$ is called a wavelet frame.

Theorem 2[5]. Let $a>1, b>0$ and $\psi\in L^2(R^2)$ be given. If $\{a^{j/2}\psi(a^j x-kb),\ k\in Z^2\}_{j\in Z}$ is a frame with frame bounds A,B, then

$$bA\le\sum_{j\,k\in Z^2}\left|\hat{\psi}(a^j\gamma)\right|^2\le bB,\quad a.e.$$

Theorem 3[5]. Let $a>1, b>0$ be two constants and $\psi\in L^2(R^2)$ be given. Suppose that

$$B:=(1/b)\sup_{|\gamma|\in[1,a]}\sum_{j\in Z}\sum_{k\in Z^2}\left|\hat{\psi}(a^j\gamma)\hat{\psi}(a^j\gamma+k/b)\right|<+\infty$$

Then the family $\{a^{j/2}\psi(a^j x-kb)\}_{j,k\in Z}$ is a Bessel sequence with bound B, and for all functions in $L^2(R^2)$ for which $\hat{f}\in C_c(R)$,

$$\sum_{j,k\in Z}\left|\langle f, D_{a^j}T_{kb}\psi\rangle\right|^2=\frac{1}{b}\int_{-\infty}^{\infty}\left|\hat{f}(r)\right|^2\sum_{j\in z}\left|\hat{\psi}(a^j r)\right|^2 d\gamma+\frac{1}{b}\sum_{k\ne 0}\sum_{j\in Z}\int_{-\infty}^{\infty}\hat{f}(r)\overline{\hat{f}(\gamma-a^j k/b)\hat{\psi}(a^{-j}\gamma)}\hat{\psi}(a^{-j}\gamma-k/b)d\gamma$$

If furthermore

$$A := (1/b)\inf_{|\gamma|\in[1,a]}(\sum_{j\in Z}\left|\hat{\psi}(a^j\gamma)\right|^2 - \sum_{k\neq 0}\sum_{j\in Z}\left|\hat{\psi}(a^j\gamma)\hat{\psi}(a^j\gamma + k/b)\right|) > 0,$$

then $\{a^{j/2}\psi(a^j x - kb)\}_{j,k\in Z}$ is a frame for $L^2(R)$ with bounds A,B.

Theorem 4[5]. Let $\psi \in L^2(R)$ and assume that there exists a constant $C > 0$ such that

$$\left|\hat{\psi}(\gamma)\right| \le C|\gamma|\cdot(1+|\gamma|^2)^{-3/2} \quad a.e.$$

Then, for all $a > 1$ and $b > 0$, it follows that

$$\sum_{k\neq 0}\sum_{j\in Z}\left|\hat{\psi}(a^j\gamma)\hat{\psi}(a^j\gamma + k/b)\right| \le 16C^2 b^{4/3}(a^2/(a-1) + a/(a^{2/3}-1))$$

Construction of tight frames for $L^2(R^2)$ and pyramid decomposition scheme

In order to split a function $\Phi(x)$ of V_1 into two functions (mostly) in V_0 and W_0, respect-ively, we will construct an affine pseudoframe for V_0 making use of the existing affine pseudoframe structure for V_0. Conven-tional symbols, $\psi_\iota(x)$ and $\widetilde{\psi}_\iota(x)$, will be used as generating functions for thesubspce $W_0 = V_1 - V_0$. But they need not be contained in W_0.

Definition 5. Let $\{V_n, \phi(x), \tilde{\phi}(x)\}$ be a given BGMS, and let $\psi_\iota(x)$ and $\widetilde{\psi}_\iota(x)$ $(\iota = 1,2,\cdots r)$ be $2r$ band-pass functions in $L^2(R^2)$. We say $\{T_v\phi(x), T_v\psi_\iota(x)\}$, $\iota \in I = \{1,2,\cdots r\}$ forms an affine ps -eudoframe for V_1 with respect to $\{T_v\tilde{\phi}(x), T_v\widetilde{\psi}_\iota(x), \iota\in I\}$, if

$$\forall\Phi \in V_1,\quad \Phi = \sum_{v\in Z^2}\left\langle \Phi, T_v\tilde{\phi}\right\rangle T_v\phi + \sum_{l\in I}\sum_{v\in Z^2}\left\langle \Phi, T_v\widetilde{\psi}_l\right\rangle T_v\psi_l, \tag{7}$$

Accordingly, $\{T_v\tilde{\phi}, T_v\widetilde{\psi}_l\}$ is called a dual affine pseudoframe to $\{T_v\phi, T_v\psi_\iota\}, \iota\in I$ in the sense of (7).

Theorem 5[6]**.** Let Y be the bandwidth of the subspace V_0 defined in Proposition 1. $\{T_v\phi, T_v\psi_l,\ v\in Z,\ l\in I\}$ forms an affine pseudoframe for V_1 with respect to $\{T_v\tilde{\phi}, T_v\widetilde{\psi}_l, l\in I\}$ if and only if there exist filter functions G_0 and $G_l, (l\in I)$ in $L^2([0,1]^2)$) such that

$$G_0(\omega)\overline{\widetilde{B}_0(\omega)}\Gamma_\Lambda(\omega) + \sum_{l=1}^{r} G_l(\omega)\overline{\widetilde{D}_l(\omega)}\Gamma_\Lambda(\omega) = (r+1)\Gamma_\Lambda(\omega), \tag{8}$$

$$\{G_0(\omega+\sigma/4)\overline{\widetilde{B}_0(\omega+\sigma/4)} + \sum_{l=1}^{r} G_l(\omega+\sigma/4)\overline{\widetilde{D}_l(\omega+\sigma/4)}\}\chi_Y(\omega) = 0,\quad \sigma = 1,2,\cdots,r. \tag{9}$$

Summary

The binary minimum-energy wavelet frames and frame multiresolution resolution are introduce--d. The sufficient condition for the existence of a class of affine pseudoframes with filter banks is obtained by virtue of a generalized multiresolution analysis.

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