On the extension of Hilbert Inequality for Finite Series

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Abstract. In this paper, by introducing parameter l, and using Hadamard inequality and Cauchy inequality, an extension of Hilbert inequality with finite version is established.

Introduction

If $a_n \ge 0, b_n \ge 0$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty, 0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then the well known Hilbert inequality and its equivalent inequalities are given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < p\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \tag{1}$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_m b_n}{m+n} < p\left(\sum_{n=1}^{N} a_n^2\right)^{1/2}\left(\sum_{n=1}^{N} b_n^2\right)^{1/2}. \tag{2}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < p\left(\sum_{n=0}^{\infty} a_n^2\right)^{1/2}\left(\sum_{n=0}^{\infty} b_n^2\right)^{1/2}. \tag{3}$$

$$\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{a_m b_n}{m+n+1} < p\left(\sum_{n=0}^{N} a_n^2\right)^{1/2}\left(\sum_{n=0}^{N} b_n^2\right)^{1/2}. \tag{4}$$

(see Hardy et al.[1]). In recently years, various improvements and extensions of the Hilbert inequality and Hilbert type inequalities appear in a great deal of papers (see [2-5]). Zhang xiaoming, Chu yuming ([2]) gave improvements of (2) ,(4)as:

$$p^2\sum_{n=1}^{N} a_n^2\sum_{n=1}^{N} b_n^2 - \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{a_m b_n}{m+n}\right)^2 \ge \min_{1\le n\le N}\left\{n a_n^2\right\}\min_{1\le n\le N}\left\{n b_n^2\right\}\left[p^2\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2 - \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{\sqrt{mn}(m+n)}\right)^2\right] \tag{5}$$

and

$$p^2\sum_{n=0}^{N} a_n^2\sum_{n=0}^{N} b_n^2 - \left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{a_m b_n}{m+n+1}\right)^2$$
$$\ge \min_{0\le n\le N}\left\{\left(n+\tfrac{1}{2}\right)a_n^2\right\}\min_{0\le n\le N}\left\{\left(n+\tfrac{1}{2}\right)b_n^2\right\}\left[p^2\left(\sum_{n=0}^{N}\frac{1}{n+\frac{1}{2}}\right)^2 - \left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{1}{\sqrt{\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}(m+n)}\right)^2\right] \tag{6}$$

The major objective of this paper is to formulate new inequalities, which are extensions of (5), (6).

Some lemmas

In order to prove our main result we need some lemmas, which we present in this section.

Lemma 1 (Hadamard inequality)[2] Let $j:[c,d] \to$ ¡ be a convex function, then one has

$$j\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d j(x)dx \le \frac{j(c)+j(d)}{2}.$$

Lemma 2 If $1 \le n \le N,\ 0 < l \le 2,$ then

$$\sum_{m=1}^{N} \frac{1}{m^{1-l/2}(m^l+n^l)} < \frac{p}{l} n^{-l/2}. \tag{7}$$

Proof. Since $\frac{1}{x^{1-l/2}(x^l+n^l)}$ is monotone decreasing in the interval $0 < x < \infty$, then

$$\sum_{m=1}^{N} \frac{1}{m^{1-l/2}(m^l+n^l)} < \int_0^N \frac{1}{x^{1-l/2}(x^l+n^l)} \mathrm{d}x = \frac{1}{l} n^{-\frac{l}{2}} \int_0^{\left(\frac{N}{n}\right)^l} \frac{1}{\sqrt{t}(1+t)} \mathrm{d}t < \frac{1}{l} n^{-\frac{l}{2}} \int_0^{\infty} \frac{1}{\sqrt{t}(1+t)} \mathrm{d}t = \frac{p}{l} n^{-\frac{l}{2}}.$$

Lemma 3 If $1 \le n \le N,\ 0 < l \le 1,$ then

$$\sum_{m=0}^{N} \frac{1}{\left(m+1/2\right)^{1-l/2}\left((m+1/2)^l+(n+1/2)^l\right)} < \frac{p}{l}\left(n+1/2\right)^{-l/2}. \tag{8}$$

Proof. Since $0 < l \le 1$, The second derivative of the function $\frac{1}{\left(x+1/2\right)^{1-l/2}\left((x+1/2)^l+(n+1/2)^l\right)}$ is positive, then the function is a convex function, by using Hadamard inequality, we have

$$\frac{1}{\left(m+1/2\right)^{1-l/2}\left((m+1/2)^l+(n+1/2)^l\right)} \le \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{1}{\left(x+1/2\right)^{1-l/2}\left((x+1/2)^l+(n+1/2)^l\right)} dx,$$

$$\sum_{m=0}^{N} \frac{1}{\left(m+1/2\right)^{1-l/2}\left((m+1/2)^l+(n+1/2)^l\right)} \le \sum_{m=0}^{N} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \frac{1}{\left(x+1/2\right)^{1-l/2}\left((x+1/2)^l+(n+1/2)^l\right)} dx$$

$$= \int_{-\frac{1}{2}}^{N+\frac{1}{2}} \frac{1}{\left(x+1/2\right)^{1-l/2}\left((x+1/2)^l+(n+1/2)^l\right)} dx = \frac{1}{l}\left(n+1/2\right)^{-\frac{l}{2}} \int_0^{\left(\frac{N+1}{n+1/2}\right)^l} \frac{1}{\sqrt{t}(1+t)} dt$$

$$< \frac{1}{l}\left(n+1/2\right)^{-\frac{l}{2}} \int_0^{\infty} \frac{1}{\sqrt{t}(1+t)} dt = \frac{p}{l}\left(n+1/2\right)^{-\frac{l}{2}}..$$

This concludes the proof.

Main results

Theorem 1 If $a_n \ge 0, b_n \ge 0, n = 1, 2, \mathbf{L}, N.\ \ 0 < l \le 2,$ then

$$\left(\frac{p}{l}\right)^2 \sum_{n=1}^{N} n^{1-l} a_n^2 \sum_{n=1}^{N} n^{1-l} b_n^2 - \left(\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{a_m b_n}{m^l+n^l}\right)^2$$

$$\ge \min_{1\le n\le N}\left\{n^{2-l}a_n^2\right\} \min_{1\le n\le N}\left\{n^{2-l}b_n^2\right\} \left[\left(\frac{p}{l}\right)^2 \left(\sum_{n=1}^{N} \frac{1}{n}\right)^2 - \left(\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{1}{(mn)^{1-l/2}(m^l+n^l)}\right)^2\right]. \tag{9}$$

Proof. Let $c_n = n^{1-\frac{l}{2}} a_n, d_n = n^{1-\frac{l}{2}} b_n,$ then inequality (9) is translated into

$$\left(\frac{p}{l}\right)^2 \sum_{n=1}^{N} \frac{c_n^2}{n} \sum_{n=1}^{N} \frac{d_n^2}{n} - \left(\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{c_m d_n}{(mn)^{1-l/2}(m^l+n^l)}\right)^2$$

$$\geq \min_{1\le n\le N}\left\{c_n^2\right\}\min_{1\le n\le N}\left\{d_n^2\right\}\left[\left(\frac{p}{l}\right)^2\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{(mn)^{1-l/2}(m^l+n^l)}\right)^2\right]. \qquad (10)$$

By using Cauchy inequality, we have

$$\left(\frac{p}{l}\right)^2\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{c_m d_n}{(mn)^{1-l/2}(m^l+n^l)}\right)^2$$

$$\geq\left(\frac{p}{l}\right)^2\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{c_m^2}{(mn)^{1-l/2}(m^l+n^l)}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{d_n^2}{(mn)^{1-l/2}(m^l+n^l)}. \qquad (11)$$

Let $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$

$$=\left(\frac{p}{l}\right)^2\sum_{n=1}^{N}\frac{c_n^2}{n}\sum_{n=1}^{N}\frac{d_n^2}{n}-\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{c_m^2}{(mn)^{1-l/2}(m^l+n^l)}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{d_n^2}{(mn)^{1-l/2}(m^l+n^l)},$$

to compute the partial derivatives of a function $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ with respect to $c_i\,.\,d_i$, and using lemma 2, we have

$$\frac{\partial f}{\partial c_l}=\left(\frac{p}{l}\right)^2\frac{2c_i}{i}\sum_{n=1}^{N}\frac{d_n^2}{n}-\sum_{n=1}^{N}\frac{2c_i}{i^{1-l/2}n^{1-l/2}(i^l+n^l)}\sum_{n=1}^{N}\frac{d_n^2}{n^{1-l/2}}\left(\sum_{m=1}^{N}\frac{1}{m^{1-l/2}(m^l+n^l)}\right)$$

$$>\left(\frac{p}{l}\right)^2\frac{2c_i}{i}\sum_{n=1}^{N}\frac{d_n^2}{n}-\frac{2c_i}{i^{1-l/2}}i^{-l/2}\frac{p}{l}\sum_{n=1}^{N}\frac{d_n^2}{n^{1-l/2}}\left(\frac{p}{l}n^{-l/2}\right)=0.$$

thus $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ is monotone increasing for c_i. In a similar way we can provde that $\frac{\partial f}{\partial d_i}>0,$ and this implies $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ is monotone increasing for d_i. We obtain

$$f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$$

$$\geq f(\min_{1\le n\le N}\{c_n\},\min_{1\le n\le N}\{c_n\},\mathbf{L},\min_{1\le n\le N}\{c_n\},\min_{1\le n\le N}\{d_n\},\min_{1\le n\le N}\{d_n\},\mathbf{L},\min_{1\le n\le N}\{d_n\}).$$

In view of (11), (10) holds. The theorem is proved.

Theorem 2 If $a_n\geq 0, b_n\geq 0, n=0,1,2,\mathbf{L},N.$ $0<l\leq 1,$ then

$$\left(\frac{p}{l}\right)^2\sum_{n=0}^{N}\left(n+\tfrac{1}{2}\right)^{1-l}a_n^2\sum_{n=0}^{N}\left(n+\tfrac{1}{2}\right)^{1-l}b_n^2-\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{a_m b_n}{(m+\frac{1}{2})^l+(n+\frac{1}{2})^l}\right)^2$$

$$\geq\min_{0\le n\le N}\left\{\left(n+\tfrac{1}{2}\right)^{2-l}a_n^2\right\}\min_{0\le n\le N}\left\{\left(n+\tfrac{1}{2}\right)^{2-l}b_n^2\right\}$$

$$\times\left[\left(\frac{p}{l}\right)^2\left(\sum_{n=0}^{N}\frac{1}{n+\frac{1}{2}}\right)^2-\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{1}{\left(n+\frac{1}{2}\right)^{1-l/2}\left(m+\frac{1}{2}\right)^{1-l/2}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\right)^2\right]. \qquad (12)$$

Proof. Let $c_n=\left(n+\frac{1}{2}\right)^{1-\frac{l}{2}}a_n, d_n=\left(n+\frac{1}{2}\right)^{1-\frac{l}{2}}b_n,$ then inequality (12) is translated into

$$\left(\frac{p}{l}\right)^2\sum_{n=0}^{N}\frac{c_n^2}{n+\frac{1}{2}}\sum_{n=0}^{N}\frac{d_n^2}{n+\frac{1}{2}}-\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{c_m d_n}{(n+\frac{1}{2})^{1-l/2}(m+\frac{1}{2})^{1-l/2}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\right)^2$$

$$\ge \min_{0\le n\le N}\left\{c_n^2\right\}\min_{0\le n\le N}\left\{d_n^2\right\}\left[B^2\left(\frac{l}{2},\frac{l}{2}\right)\left(\frac{p}{l}\right)^2-\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{1}{(n+\frac{1}{2})^{1-\frac{l}{2}}(m+\frac{1}{2})^{1-\frac{l}{2}}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\right)^2\right]. \tag{13}$$

By using Cauchy inequality, we have

$$\left(\frac{p}{l}\right)^2\sum_{n=0}^{N}\frac{c_n^2}{n+\frac{1}{2}}\sum_{n=0}^{N}\frac{d_n^2}{n+\frac{1}{2}}-\left(\sum_{n=0}^{N}\sum_{m=0}^{N}\frac{c_m d_n}{(n+\frac{1}{2})^{1-\frac{l}{2}}(m+\frac{1}{2})^{1-\frac{l}{2}}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\right)^2$$

$$>\left(\frac{p}{l}\right)^2\sum_{n=0}^{N}\frac{c_n^2}{n+\frac{1}{2}}\sum_{n=0}^{N}\frac{d_n^2}{n+\frac{1}{2}}$$

$$-\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{c_m^2}{(n+\frac{1}{2})^{1-\frac{l}{2}}(m+\frac{1}{2})^{1-\frac{l}{2}}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{d_n^2}{(n+\frac{1}{2})^{1-\frac{l}{2}}(m+\frac{1}{2})^{1-\frac{l}{2}}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)} \tag{14}$$

To define a function $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ by the right hand side of (14), computting the partial derivatives of a function $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ with respect to $c_i\,.\,d_i$, and using lemma 3, we have

$$\frac{\partial f}{\partial c_l}=\left(\frac{p}{l}\right)^2\frac{2c_i}{i+\frac{1}{2}}\sum_{n=0}^{N}\frac{d_n^2}{n+\frac{1}{2}}$$

$$-\sum_{n=0}^{N}\frac{2c_i}{\left(i+\frac{1}{2}\right)^{1-\frac{l}{2}}\left(n+\frac{1}{2}\right)^{1-\frac{l}{2}}\left((i+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\sum_{n=1}^{N}\frac{d_n^2}{\left(n+\frac{1}{2}\right)^{1-\frac{l}{2}}}\left(\sum_{m=0}^{N}\frac{1}{\left(m+\frac{1}{2}\right)^{1-\frac{l}{2}}\left((m+\frac{1}{2})^l+(n+\frac{1}{2})^l\right)}\right)$$

$$>\left(\frac{p}{l}\right)^2\frac{2c_i}{i+\frac{1}{2}}\sum_{n=0}^{N}\frac{d_n^2}{n+\frac{1}{2}}-\frac{2c_i}{\left(i+\frac{1}{2}\right)^{1-\frac{l}{2}}}\left(i+\frac{1}{2}\right)^{-\frac{l}{2}}\frac{p}{l}\sum_{n=0}^{N}\frac{d_n^2}{\left(n+\frac{1}{2}\right)^{1-\frac{l}{2}}}\left(\frac{p}{l}\left(n+\frac{1}{2}\right)^{-\frac{l}{2}}\right)=0,$$

and $\dfrac{\partial f}{\partial d_l}>0$, thus $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)$ is monotone increasing for $c_i\,,\,d_i$. We get $f(c_1,c_2,\mathbf{L},c_n,d_1,d_2,\mathbf{L},d_n)\ge f(\min_{1\le n\le N}\{c_n\},\mathbf{L},\min_{1\le n\le N}\{c_n\},\min_{1\le n\le N}\{d_n\},\mathbf{L},\min_{1\le n\le N}\{d_n\})$. In view of (14), (13) holds. The theorem is proved.

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