Some Results for the Existence of Periodic Solutions to $p$-Laplacian Equation on Time Scales

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Abstract—In this paper, we investigate a class of Liénard type $p$-Laplacian equation on times scales by generalized Mawhin’s continuation theorems, under suitable conditions, we ensure that at least one periodic solution to this kind of $p$-Laplacian equation on time scales exist.

Keywords—generalized mawhin’s continuation theorems; $p$-Laplacian; periodic solution; time scales.

I. INTRODUCTION

Liénard equations can be derived from many fields, such as mechanics, engineering technique fields, physics, and so on, and is important in describing fluid mechanical and nonlinear elastic mechanical phenomena.

Many authors have contributed to the theory of the equations with respect to existence of periodic solutions (see e.g. [1–6] and the reference therein), during the past several years.

The important and useful tools to study this class of differential equations are Mawhin’s continuation theorem, generalized polar coordinates, Leary-Schauder degree theory and many fixed point theory.

Mawhin’s continuation theorems has been extensively used for getting the existence of periodic solutions to this class equation.

For example, using Mawhin’s continuation theorem, Cheung and Ren considered the existence of $T$-periodic solutions to a Liénard type $p$-Laplacian equation with a deviating argument in [7],

\[(\varphi_p(x'(t)))' + F(x(t))x'(t) + G(x(t - \tau(t))) = E(t),\]

and some results for the existence of periodic solutions were got. Lu investigated the existence of periodic solutions for a $p$-Laplacian Liénard differential equation with a deviating argument by using Mawhin’s continuation theorem in [8].

Du and Zhao [9] introduce us the existence of periodic solution to a $p$-Laplacian Liénard equation by means of generalized Mawhin’s continuation theorem.

In [11], Li and Zhang considered the periodic solutions for a periodic mutualism model on a time scale $T$ by employing Mawhin’s continuation theorem, and obtained three sufficient criteria.

In this paper, we will systematically investigate the existence of periodic solutions of the Liénard type $p$-Laplacian equation

\[(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t - \tau(t))) = e(t)\]

on a time scales $T$. Our technique is motivated by that used in [14], and we applying it to investigate the existence of periodic solutions for (1.1).

The setup of this paper is as following. In the coming section, we present some lemmas and definitions on time scales. In Section 3, we systematically explore the existence of periodic solutions of the Liénard type $p$-Laplacian equation on time scales.

II. PRELIMINARY

In this section, we briefly give some basic definitions, lemmas on time scales which are used in the follows. Let $T$ be a time scale (a nonempty closed subset of $\mathbb{R}$). The forward and backward jump operators $\sigma, \rho : T \to T$ and the graininess $\mu : T \to R^*_+.$

Definition 2.1. ([15]) Let $X$ and $Z$ be two Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_Z,$ respectively. A continuous operator $M : X \cap \text{dom}M \to Z$ is said to be quasi-linear if

(i) $\text{Im}M := M(X \cap \text{dom}M)$ is a closed subset of $Z$;

(ii) $\text{Ker}M := \{x \in X \cap \text{dom}M : Mx = 0\}$ is linearly homeomorphic to $R^n, \ n < \infty.$

Definition 2.2. ([15]) Let $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$. $N_x : \overline{\Omega} \to Z, \ \lambda \in [0, 1]$ is said to be $M - \text{compact in } \overline{\Omega}$ if there exists subset $Z_1$ of $Z$ satisfying $\dim Z_1 = \dim \text{Ker}M$ and an...
operator \( R : \Omega \times [0,1] \to X_2 \) being continuous and compact such that for \( \lambda \in [0,1] \),

(a) \((I - Q)N_\lambda(\Omega) \subset \text{Im} M \subset (I - Q)Z\),

(b) \( QN_\lambda x = 0, \lambda \in (0,1) \Leftrightarrow QNx = 0 \),

(c) \( R(\cdot,0) = 0 \) and \( R(\cdot,\lambda) \mid_{\{\lambda\}} = (I - P) \mid_{\{\lambda\}} \),

(d) \( M(P + R(\cdot,\lambda)] = (I - Q)N_\lambda, \lambda \in [0,1] \),

where \( X_2 \) is the complement space of \( \text{Ker} M \) in \( X \), i.e.,

\[ X = \text{Ker} M \oplus X_2, P,Q \text{ are two projectors satisfying} \]

\[ \text{Im} P = \text{Ker} M, \quad \text{Im} Q = Z_1, \quad N = N_1, \]

\[ \Sigma_\lambda = \{ x \in \Omega : Mx = N_\lambda x \} \].

**Lemma 2.1.** ([15]) Let \( X \) and \( Z \) be two Banach spaces with norms \( \| \cdot \|_X, \| \cdot \|_Z \) respectively and \( \Omega \subset X \) be an open and bounded nonempty set. Suppose \( M : X \cap \text{dom} M \to Z \) is quasi-linear and \( N_\lambda : \Omega \to Z, \lambda \in [0,1] \) is \( M \)-compact in \( \Omega \). In addition, if the following conditions hold:

- For \( (H_1) \quad ) \begin{align*} \text{Q}\text{N} x & \neq 0, \forall x \in \text{Ker} M \cap \partial \Omega \ , \ (H_2) \quad \text{deg} \{\text{J}\text{Q}\text{N}, \Omega \rightleftharpoons \text{Ker} M, 0\} \neq 0, \end{align*} \]

- For any bounded set \( \text{dom} M \subset \Omega \subset X \) define \( M = \text{Ker} M \sqcup \text{dom} M \sqcup Z_1 \) a quasi-linear operator. For all \( t \in T \), define the operator \( P,Q \) by

\[ P : X \to \text{Ker} M, (Pc)(t) = x(0), Q : Z \to R(QZ)(t) = \int_0^T z(s)ds \]

then \( \lambda \) is a homeomorphism. Then the abstract equation \( Mx = Nx \) has at least one solution in \( \text{dom} M \sqcup \Omega \).

**III. MAIN RESULTS**

For convenience of applying Lemma 2.3, we denote

\[ X = C_1^1 = \{ x \mid x \in C_1^1(T,R), x(t+T) = x(t), x^\lambda(t+T) = x^\lambda(t) \} \]

\[ Z = C_\lambda = \{ x \mid x \in C_\lambda(T,R), x(t+T) = x(t) \} \]

\[ \| x \| = \max \{ \| x \|_X, \| x^\lambda(t) \|_Z, \| x^\lambda(t) \|_Z \} , \| x \| = \max \{ \| x \|_X, \| x^\lambda(t) \|_Z \} \in [0,T] \cap T \]

where \( [0,T]_R \) denote the interval \([0,T]\) on \( R \). Then the operators \( M, N_\lambda \) are defined by

\[ M : \text{dom} M \cap X \to Z, (Mx)(t) = (\varphi_x(x^\lambda(t)))^3 \], \hspace{1cm} (2)

\[ N_\lambda : X \to Z(N_\lambda x)(s) = -\lambda(f(x(s))x^\lambda(s) - g(x(s) - \varphi(s))) + \lambda \varphi(s), \lambda \in [0,1] \] \hspace{1cm} (3)

where \( \text{dom} M = \{ x \mid \varphi_x(x^\lambda(t)) \in C_1^1 \} \); \( f,g \in C(R,R); e, \varphi \in C(T,R) \), \( e(t+T) = e(t), \varphi(t+T) = \varphi(t) \). Let

\[ F(t,x) = \varphi_x(x^\lambda(t)) \rightarrow f(x(t)x^\lambda(t) - g(x(t) - \varphi(t))) + \varphi(t) \] \hspace{1cm} (4)

then \( N_\lambda x = \lambda F \). By (2) and (3), Eq. (1) is equivalent to the operator equation \( Mx = Nx \), where \( N_1 = N \). Then we have

\[ \text{Ker} M = \{ x \in X \mid x = a \in R \} \approx R \]

\[ \text{Im} M = \{ z \in Z \mid \int_0^T z(s)ds = 0 \} \]

Then we have the following Lemma.

**Lemma 3.1.** Let \( M \) be as defined by (2). Then \( M \) is a quasi-linear operator. For all \( t \in T \), define the operator \( P,Q \) by

\[ P : X \to \text{Ker} M, (Pc)(t) = x(0), Q : Z \to R(QZ)(t) = \int_0^T z(s)ds \]

\[ \frac{(c)}{\text{deg} \{\text{J}\text{Q}\text{N}, \Omega \rightleftharpoons \text{Ker} M, 0\} \neq 0, \end{align*} \]

\[ J : \text{Im} Q \to \text{Ker} M \] is a homeomorphism. Then the abstract equation \( Mx = Nx \) has at least one solution in \( \text{dom} M \sqcup \Omega \).

**Step 4.** Since \( Q^2 = Q \), we have

\[ Q(I - Q)N_\lambda(\Omega) = 0, \quad \text{so} \]

\[ (I - Q)N_\lambda(\Omega) \subset \text{Ker} Q = \text{Im} M \]

On the other hand, \( \forall z \in \text{Im} M \), clearly, \( Qz = 0 \), so \( z = z - Qz = (I - Q)z \), then \( z \in (I - Q)Z \). So we have

\[ (I - Q)N_\lambda(\Omega) \subset \text{Im} M \subset (I - Q)Z \]

**Step 2.** We show that

\[ QN_\lambda x = 0, \lambda \in (0,1) \Leftrightarrow QNx = 0, \forall x \in \Omega \]
Step 3. When \( \lambda = 0 \), since \( a_\varepsilon \in [-A,-B] \), then there exist \( a_\varepsilon = 0 \). For \( a_\varepsilon = 0 \), we have \( R(x,0)(t) \equiv 0 \).

\[
\forall x \in \Sigma_1 = \{x \in \overline{\Omega} : Mx = N_\varepsilon x\},
\]
we have \( (\varphi_\varepsilon(x^\varepsilon(t)))^\lambda = \lambda F \) and \( QF = \int_0^\infty (\varphi_\varepsilon(x^\varepsilon(t)))^\lambda \Delta t = 0 \).

For

\[
R(x,\lambda)(t) = \int_0^t \varphi_\varepsilon[a_\varepsilon + \int_0^t \lambda(F - QF)(r)\Delta r] \Delta s,
\]
take \( a_\varepsilon = -\varphi_\varepsilon(x^\varepsilon(0)) \), we obtain

\[
R(x,\lambda)(t) = \int_0^t \varphi_\varepsilon[-\varphi_\varepsilon(x^\varepsilon(0)) + \int_0^t \lambda(F - QF)(r)\Delta r] \Delta s = \int_0^t \varphi_\varepsilon[-\varphi_\varepsilon(x^\varepsilon(0)) + \int_0^t \lambda(F - QF)(r)\Delta r] \Delta s = \lambda(F - QF)(t) = (N_\varepsilon - QN_\varepsilon)x(t).
\]

Step 4. \( \forall x \in \overline{\Omega} \), we have

\[
M[\overline{P}x + R(x,\lambda)](t) = (\varphi_\varepsilon(x(t)) + \int_0^t \varphi_\varepsilon[-\varphi_\varepsilon(x^\varepsilon(0)) + \int_0^t \lambda(F - QF)(r)\Delta r]]^\lambda = (\varphi_\varepsilon[-\varphi_\varepsilon(x^\varepsilon(0)) + \int_0^t \lambda(F - QF)(r)\Delta r]^{\lambda} = \lambda(F - QF)(t) = (N_\varepsilon - QN_\varepsilon)x(t).
\]

Hence, \( N_\varepsilon \) is \( M \)-compact in \( \overline{\Omega} \).

Theorem 3.1. Suppose \( f, g \in C(R,R) \),

e, \tau(t) \in C(T,R)

with

\( e(t) = e(t + T) \) and \( \tau(t) = \tau(t + T) \), there exist constants \( d_1 \), assume that the following conditions

(i) \( f(u(t))u^\lambda(t) > 0 \), when \( |u(t - \tau(t))| \geq d_1 \),

(ii) \( \lim_{|u| \to \infty} \frac{g(u)}{|u|^{\lambda+1}} = r > 0 \),

(iii) \( \lambda T \max_{|u| \to \infty} |f(u)| \leq 1 \), if \( p = 2 \). Then Eq. (1) has at least one \( T \)-periodic solution.

Proof We complete the proof by three steps.

Step 1. Let

\( \Omega_4 = \{x \in domM : Mx = N_\varepsilon x, \lambda \in (0,1)\} \). We claim that \( \Omega_4 \) is a bounded set. If \( x \in \Omega_4 \), then \( Mx = N_\varepsilon x \), i.e.,

\[
(\varphi_\varepsilon(x^\varepsilon(t)))^\lambda + \lambda f(x(t))x^\lambda(t) + \lambda g(x(t - \tau(t))) = \lambda e(t).
\]

Integrating both sides of (5) over \([0,T]\), we have

\[
\int_0^T f(x(s))x^\lambda(s) \Delta s = \int_0^T g(x(s - \tau(s))) \Delta s + \int_0^T e(s) \Delta s,
\]

that is,

\[
\int_0^T f(x(s))x^\lambda(s) \Delta s = -\int_0^T g(x(s - \tau(s))) - e(s) \Delta s.
\]

Then we have

\[
\int_0^T \left[ f(x(s))x^\lambda(s) + g(x(s - \tau(s))) - e(s) \right] \Delta s = 0.
\]

There must exist some \( \xi \) such that

\[
f(x(\xi))x^\lambda(\xi) + g(x(\xi - \tau(\xi))) - e(\xi) \leq 0.
\]

From the assumption (i) and (ii), we have

\[
f(x(\xi))x^\lambda(\xi) > 0, \quad |x(\xi - \tau(\xi))| \geq d_1,
\]

and there exist constants \( d_2 > 0 \) and \( \varepsilon > 0 \) such that

\[
g(x(\xi - \tau(\xi))) - e(\xi) = (\varepsilon + \varepsilon) |x(\xi - \tau(\xi))| > 0, \quad |x(\xi - \tau(\xi))| \geq d_2.
\]

So when \( |x(\xi - \tau(\xi))| \geq d_2 \), we obtain

\[
f(x(\xi))x^\lambda(\xi) \leq -g(x(\xi - \tau(\xi))) + e(\xi) \leq -g(x(\xi - \tau(\xi)))+|e| \leq 0.
\]

Then we get

\[
|x(\xi - \tau(\xi))| \leq \max\{d_1, d_2\}, \quad \text{and} \quad |\tau(t)| \leq |x(\xi - \tau(\xi))| + \int_0^t \left| x^\lambda(s) \right| \Delta s \leq \max\{d_1, d_2\} + \int_0^t \left| x^\lambda(s) \right| \Delta s.
\]

Take the absolute value of both sides of the equation (3.4), and integrating it over \([0, T]\),

\[
\int_0^T |x^\lambda(s)|^{-1} \Delta s \leq \lambda \int_0^T |f(x(s))x^\lambda(s)| |x^\lambda(s)|^{-1} \Delta s + \lambda \int_0^T |g(x(s - \tau(s))) - e(s)| \Delta s
\]

\[
\leq \lambda \max_{\|u\| \to \infty} |f(u)| \int_0^T |x^\lambda(s)|^{-1} \Delta s + \lambda T \max_{\|u\| \to \infty} |g(x(t - \tau(t)) - e(t)|.
\]

By Hölder inequality, and combining (7) we have

\[
\int_0^T |x^\lambda(s)|^{-1} \Delta s \leq \left( \int_0^T |x^\lambda(s)|^{-1} \Delta s \right)^{\frac{1}{p'}} T \frac{p}{p'}.
\]

Substituting above inequality into (8) we obtain

\[
\int_0^T |x^\lambda(s)|^{-1} \Delta s \leq \lambda T \max_{\|u\| \to \infty} |f(x(s))| \left( \int_0^T |x^\lambda(s)|^{-1} \Delta s \right)^{\frac{1}{p'}} + \lambda T \max_{\|u\| \to \infty} |g(x(t - \tau(t)) - e(t)|.
\]

Since \( \frac{1}{p'} \leq 1 \), so \( |x^\lambda(t)| \) bounded, that means there exist \( M_2 \) such that \( |x^\lambda(t)| \leq M_2 \), so we have

\[
|x(t)| \leq \max\{d_1, d_2\} + TM_2 := M_0.
\]

Step 2 Let \( \Omega_2 = \{x \in KerM : QN_\varepsilon x = 0\} \). For \( \forall x \in \Omega_2 \), then

\[
x(t) = a_0 \in R.
\]

Since

\[
QN_\varepsilon x = -f(x(t))x^\lambda(t) - g(x(t - \tau(t))) + e(t),
\]
we have

\[
QN_\varepsilon x = -f(x(t))x^\lambda(t) - g(x(t - \tau(t))) + e(t) = -g(u_a) + \frac{1}{T} \int_0^T e(s) \Delta s = 0.
\]

From the assumption, \( \lim_{|u| \to \infty} \frac{g(u)}{|u|^{\lambda+1}} = r > 0 \), we have

\[
|a_0| \leq d_2.
\]

Take the open and bounded set
\[ \Omega \supseteq \Omega_1 \cup \Omega_2, \text{ then the conditions } (H_1) \text{ and } (H_2) \text{ of Lemma 2.3 satisfied.} \]

**Step 3** Define the operator \( J : \text{Im}Q \to \text{Ker}M, J(a) = a, a \in R. \) Take

\[ H(x, \mu) = \mu a_0 - (1 - \mu)JQN, \]

then

\[ H(a_0, \mu) = a_0 \mu - (1 - \mu)(-g(a_0) + \int_0^\tau e(s) \Delta s), \]

and

\[ a_0 H(x, \mu) = \mu a_0^2 - (1 - \mu) a_0 (-g(a_0) + \int_0^\tau e(s) \Delta s) > 0. \]

So

\[ H(a_0, \mu) \neq 0. \]

\[ \text{deg} [JQN, \Omega \cap \text{Ker}M, 0] = \text{deg} [-I, \Omega \cap \text{Ker}M, 0] \neq 0. \]

From the above prove, we can get the fact that Eq. (1) has at least one \( T \)-periodic solution. The proof is completed.

**Theorem 2.** Assume the condition (ii) of Theorem 3.1 holds, and the following conditions satisfied:

(iv) there exists a continuous function \( c(t) \) on time scales \( T \) satisfies

\[ -f(u(t))u^\Delta(t) - g(u(t - \tau(t))) + e(t) \geq c(t). \]

(v) there exists a constant \( R_1 > 0 \) such that

\[ \int_0^\tau [-f(u(t))u^\Delta(t) - g(u(t - \tau(t))) + e(t)] \Delta t > 0, \quad u_\tau \geq R_1 \| u^\Delta(t) \| M, \]

and

\[ \int_0^\tau [-f(u(t))u^\Delta(t) - g(u(t - \tau(t))) + e(t)] \Delta t < 0, \quad u_\tau \leq -R_1 \| u^\Delta(t) \| M. \]

Then Eq. (1) has at least one \( T \)-periodic solution. Where

\[ u_L := \min_{r \in [0,T]} u(t), \quad u_M := \max_{r \in [0,T]} u(t). \]

**Proof Step 1.** Let \( \Omega_1 = \{ x \in \text{dom}M : Mx = N_x, \lambda \in (0,1) \} \). We show that \( \Omega_1 \) is a bounded set. If \( x \in \Omega_1 \), then \( Mx = N_x, \)

\[ (\varphi_x(x^\Delta(t)))^\Delta + \lambda f(x(t))x^\Delta(t) + \lambda g(x(t - \tau(t))) = \lambda e(t). \]

From the definition of operator \( Q \), we know that

\[ QMx(t) = QN_x(t) = 0, \quad \text{and} \quad QNx(t) = 0. \]

The operator \( Nx \) is bounded from below by \( c \) on \( T \), so we have

\[ \text{the inequality: } \| Nx(t) \| \leq \| Nx(t) + 2c^-(t) \|, \forall t \in [0,T], \]

where we denote \( c^-(t) = \max\{ -c(t), 0 \} \). Combining the condition (iv) we obtain

\[ \int_0^\tau (\varphi_x(x^\Delta(t)))^\Delta | \Delta t = \lambda \int_0^\tau | N_x(t) | \Delta t \leq \int_0^\tau | N_x(t) | \Delta t + 2T | c^-(t) \|_0 = 2T | c^- |_0, \]

that is

\[ \int_0^\tau | x^\Delta(t) | \Delta t \leq 2T \| c^- \|_0, \]

then there exist a constant \( M_2 \) such that \( | x^\Delta(t) | \leq M_2 \). From the condition (v), if \( x_L \leq R_1 ( \text{or} x_M \geq R_1 ) \), we know

\[ \int_0^\tau N_x(t) \Delta t < 0 ( \text{or} \int_0^\tau N_x(t) \Delta t > 0), \]

so

\[ x_M > -R_1 ( \text{or} x_L < R_1 ). \]

Clearly, we have \( x_M \leq x_L + \int_0^\tau | x^\Delta(t) | \Delta t \). We can get

\[ -(R_1 + M_2 T) < x_L < x_M < R_1 + M_2 T, \]

that means \( x_0 < R_1 + M_2 T \). The next two steps are similar to the proof of Theorem 3.1, and then we can obtain Eq. (1) has at least one \( T \)-periodic solution. The proof is completed.

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