Ranking of fuzzy numbers based on *alpha*-distance

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Abstract

This paper presents a new approach to compare fuzzy numbers using α -distance. Initially, the metric distance on the interval numbers based on the convex hull of the endpoints is proposed and it is extended to fuzzy numbers. All the properties of the α -distance are proved in details. Finally, the ranking of fuzzy numbers by the α -distance is discussed. In addition, the proposed method is compared with some known ones, the validity of the new method is illustrated by applying its to several group of fuzzy numbers.

Keywords: Metric Distance; Fuzzy Numbers; Ranking.

1. Introduction

The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [3]. Additional related material can be found in [1, 10, 11]. Many researchers have developed methods to compare and to rank fuzzy numbers, too [9, 13, 14, 16, 19]. Cheng [4] has proposed some methods, and recently many ranking methods have been presented by Ma, Kandel and Friedman [12] and Chu and Tsao [15]. In [18] a fuzzy distance measure for Gaussian type fuzzy numbers is defined. In this study a new metric distance on fuzzy numbers is introduced and then it is used for ranking fuzzy numbers by comparison with two crisp numbers $\max(M)$ and $\min(m)$.

The rest of this paper is organized as follows: Section 2 contains preliminaries on fuzzy concepts. A metric distance measure for interval numbers with its properties is introduced in Section 3. Then, in Section 4, α -distance for fuzzy numbers is defined and its properties are discussed in detail. In the Section 5, we use a procedure for ranking fuzzy numbers based on the α -distance. For comparing the proposed ranking method with some other approaches, some numerical examples are provided in Section 6. Finally, the paper ends with conclusions in Section 7.

2. Preliminaries

A fuzzy set $\tilde{A} = (a_1, a_2, a_3, a_4)$ is a generalized left right fuzzy numbers (GLRFN) of Dubois and Prade [8, 16], if its membership function satisfies the following:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a_2-x}{a_2-a_1}), & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ R(\frac{x-a_3}{a_4-a_3}), & a_3 \leq x \leq a_4, \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

where L and R are strictly decreasing functions defined on $[0, 1]$ and satisfying the conditions:

$$\begin{aligned} L(t) = R(t) &= 1 & \text{if } t \leq 0 \\ L(t) = R(t) &= 0 & \text{if } t \geq 1 \end{aligned} \quad (2)$$

For $a_2 = a_3$, we have the classical definition of left right fuzzy numbers (LRFN) of Dubois and Prade [8, 16]. Trapezoidal fuzzy numbers (TrFN) are special cases of GLRFN with $L(t) = R(t) = 1 - t$. Triangular fuzzy numbers (TFN) are also special cases of GLRFN with $L(t) = R(t) = 1 - t$ and $a_2 = a_3$. A GLRFN $\tilde{A}$ is denoted as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$ and an α -level interval of fuzzy number $\tilde{A}$ as:

$$\begin{aligned} [\tilde{A}]^\alpha &= [A_L(\alpha), A_R(\alpha)] \\ &= [a_2 - (a_2 - a_1)L_A^{-1}(\alpha), a_3 + (a_4 - a_3)R_A^{-1}(\alpha)] \end{aligned} \quad (3)$$

If $\tilde{A}$ and $\tilde{B}$ are two GLRFN then:

$$\tilde{A} + \tilde{B} = \tilde{C} \quad \text{if and only if} \quad [\tilde{A}]^\alpha + [\tilde{B}]^\alpha = [\tilde{C}]^\alpha \quad (4)$$

and

$$\tilde{A} \cdot \tilde{B} = \tilde{C} \quad \text{if and only if} \quad [\tilde{A}]^\alpha \cdot [\tilde{B}]^\alpha = [\tilde{C}]^\alpha \quad (5)$$

where for interval numbers, we have

$$[a, b] + [c, d] = [a + c, b + d] \quad (6)$$

and

$$[a, b] \cdot [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \quad (7)$$

3. Distance for intervals

Let $f(x) = (b - a)x + a$ and $g(x) = (d - c)x + c$ where $x \in [0, 1]$. The distance between two intervals $[a, b]$ and $[c, d]$, $a \leq b, c \leq d$, is denoted by $d_I^{(p)}([a, b], [c, d])$ such that:

$$d_I^{(p)}([a, b], [c, d]) = (D_I^{(p)}([a, b], [c, d]))^{\frac{1}{p}} \quad (8)$$

and

$$D_I^{(p)}([a, b], [c, d]) = \|f(x) - g(x)\|_{L_p}^p \quad (9)$$

where $\| \cdot \|$ is the usual norm in the L_p space on $[0, 1]$ ($p > 1$).

This distance takes into account every point in both intervals with the convex combination of the lower and upper bound values of the both intervals. It is different from most of the existing distance measures for interval numbers (Bardossy et al. [1], Diamond [5], Diamond and Korner [6], and Diamond and Tanaka [7]). Tran and Duckstein [16] proposed a distance measure for intervals which also considers every point of the two intervals. But it was not a metric distance because $d([a, b], [a, b]) \neq 0$. Now we show that our proposed distance is a metric distance on interval numbers.

3.1. Metric properties

This proposed distance measure satisfies the following metric properties:

1. $d_I^{(p)}([a, b], [c, d]) \geq 0$
2. $[a, b] = [c, d]$ if and only if $d_I^{(p)}([a, b], [c, d]) = 0$
3. $d_I^{(p)}([a, b], [c, d]) = d_I^{(p)}([c, d], [a, b])$
4. $d_I^{(p)}([a, b], [c, d]) + d_I^{(p)}([c, d], [e, z]) \geq d_I^{(p)}([a, b], [e, z])$

The proofs follow immediately from the properties of the L_p -norm.

3.2. Other properties of $d_I^{(p)}$

This distance satisfies the following properties if a, b, c, d, e and z are real numbers:

Proposition 1. If $\lambda \geq 0$, then

$$d_I^{(p)}([\lambda a, \lambda b], [\lambda c, \lambda d]) = |\lambda| d_I^{(p)}([a, b], [c, d])$$

Proposition 2.

$$d_I^{(p)}([a + \lambda, b + \lambda], [c + \lambda, d + \lambda]) = d_I^{(p)}([a, b], [c, d])$$

Proposition 3.

$$d_I^{(p)}(a, b) = |a - b|$$

(both intervals collapse to a single point).

Proposition 4. If $p = 2$ then

$$d_I^{(p)}([a, b], [c, d]) = \sqrt{\frac{1}{3}((a - c)^2 + (b - d)^2 + (a - c)(b - d))}$$

Proposition 5. If $p = 2$ then

$$d_I^{(p)}([a, b], c) = \sqrt{\frac{1}{3}((a - c)^2 + (b - c)^2 + (a - c)(b - c))}$$

(one of the intervals collapses to a single point).

Proposition 6. $\frac{a+b}{2}$ is the nearest number to $[a, b]$ by using this metric.

Proposition 7. If $p = 2$ then

$$d_I^{(p)}([a, b], 0) = \sqrt{\frac{1}{3}(a^2 + b^2 + ab)}$$

is equal to the distance measure of Tran and Duckstein [16].

The proofs of Propositions 1, 2, 3, 4, 5, 6 and 7 are clear.

Proposition 8.

$$d_I^{(p)}([a, b] + [c, d], [a, b] + [e, z]) = d_I^{(p)}([c, d], [e, z])$$

Proof:

$$\begin{aligned} & d_I^{(p)}([a, b] + [c, d], [a, b] + [e, z]) \\ &= d_I^{(p)}([a + c, b + d], [a + e, b + z]) \\ &= \left(\int_0^1 ((b + d - a - c)x + a + c) \right. \\ &\quad \left. - ((b + z - a - e)x + a + e)^p dx \right)^{\frac{1}{p}} \quad (10) \\ &= \left(\int_0^1 ((d - c)x + c) - ((z - e)x + e)^p dx \right)^{\frac{1}{p}} \\ &= d_I^{(p)}([c, d], [e, z]) \end{aligned}$$

Definition 1. If $[a, b]$ and $[c, d]$ are two intervals and M is a crisp number, then by (8) and (9), we have

$$d_I^{(p)}([a, b], [c, d], M) = (D_I^{(p)}([a, b], [c, d], M))^{\frac{1}{p}} \quad (11)$$

where

$$D_I^{(p)}([a, b], [c, d], M) = \int_0^1 D_I^{(p)}([a, b], ((d - c)x + c), M) dx \quad (12)$$

If $c = d = 1$ then

$$d_I^{(p)}([a, b], [c, d], M) = d_I^{(p)}([a, b], M)$$

4. Metric for fuzzy numbers

α -distance between two GLRFNs $\tilde{A}$ and $\tilde{B}$ by using the metric defined in Section 3, can be defined as:

$$d_F^{(p)}(\tilde{A}, \tilde{B}) = (D_F^{(p)}(\tilde{A}, \tilde{B}))^{\frac{1}{p}} \quad (13)$$

such that

$$D_F^{(p)}(\tilde{A}, \tilde{B}) = \sup_{\alpha \in [0, 1]} D_I^{(p)}([\tilde{A}]^\alpha, [\tilde{B}]^\alpha) \quad (14)$$

It can be proved that $d_F^{(p)}(\tilde{A}, \tilde{B})$ is a metric on GLRFNs.

4.1. The properties of α -distance

The proposed distance ' α -distance' has the following metric properties:

1. $d_F^{(p)}(\tilde{A}, \tilde{B}) \geq 0$

2. $\tilde{A} = \tilde{B}$ if and only if $d_F^{(p)}(\tilde{A}, \tilde{B}) = 0$
3. $d_F^{(p)}(\tilde{A}, \tilde{B}) = d_F^{(p)}(\tilde{B}, \tilde{A})$
4. $d_F^{(p)}(\tilde{A}, \tilde{B}) + d_F^{(p)}(\tilde{B}, \tilde{C}) \geq d_F^{(p)}(\tilde{A}, \tilde{C})$

Proofs:

1. Since $D_I^{(p)}([\tilde{A}]^\alpha, [\tilde{B}]^\alpha) \geq 0$ then $D_F^{(p)}(\tilde{A}, \tilde{B}) \geq 0$ hence $d_F^{(p)}(\tilde{A}, \tilde{B}) \geq 0$.
2. We know that, $\tilde{A} = \tilde{B}$ if and only if $[\tilde{A}]^\alpha = [\tilde{B}]^\alpha$ for all $\alpha \in [0, 1]$. But, since $d_I^{(p)}([\cdot, \cdot], [\cdot, \cdot])$ is a metric distance, then $[\tilde{A}]^\alpha = [\tilde{B}]^\alpha$ if and only if $D_I^{(p)}([\tilde{A}]^\alpha, [\tilde{B}]^\alpha) = 0$ for all $\alpha \in [0, 1]$. Hence, $\tilde{A} = \tilde{B}$ if and only if $D_F^{(p)}(\tilde{A}, \tilde{B}) = 0$ and then $d_F^{(p)}(\tilde{A}, \tilde{B}) = 0$.
3. From equations (13) and (14),

$$\begin{aligned} d_F^{(p)}(\tilde{A}, \tilde{B}) &= \left(\sup_{\alpha \in [0, 1]} D_I^{(p)}([\tilde{A}]^\alpha, [\tilde{B}]^\alpha) \right)^{\frac{1}{p}} \\ &= \left(\sup_{\alpha \in [0, 1]} D_I^{(p)}([\tilde{B}]^\alpha, [\tilde{A}]^\alpha) \right)^{\frac{1}{p}} \\ &= \left(D_F^{(p)}(\tilde{B}, \tilde{A}) \right)^{\frac{1}{p}} \\ &= d_F^{(p)}(\tilde{B}, \tilde{A}). \end{aligned} \quad (15)$$

4. From equations (13) and (14),

$$\begin{aligned} d_F^{(p)}(\tilde{A}, \tilde{C}) &= \left(D_F^{(p)}(\tilde{A}, \tilde{C}) \right)^{\frac{1}{p}} \\ &= \left(\sup_{\alpha \in [0, 1]} \left(D_I^{(p)}([A]^\alpha, [C]^\alpha) \right) \right)^{\frac{1}{p}} \\ &= \sup_{\alpha \in [0, 1]} \left(D_I^{(p)}([A]^\alpha, [C]^\alpha) \right)^{\frac{1}{p}} \\ &= \sup_{\alpha \in [0, 1]} \left(d_I^{(p)}([A]^\alpha, [C]^\alpha) \right) \\ &\leq \sup_{\alpha \in [0, 1]} \left(d_I^{(p)}([A]^\alpha, [B]^\alpha) \right. \\ &\quad \left. + d_I^{(p)}([B]^\alpha, [C]^\alpha) \right) \\ &\leq \sup_{\alpha \in [0, 1]} \left(d_I^{(p)}([A]^\alpha, [B]^\alpha) \right) \\ &\quad + \sup_{\alpha \in [0, 1]} \left(d_I^{(p)}([B]^\alpha, [C]^\alpha) \right) \\ &= d_F^{(p)}(\tilde{A}, \tilde{B}) + d_F^{(p)}(\tilde{B}, \tilde{C}) \end{aligned}$$

Therefore, the proofs are completed.

4.2. Other properties of α -distance

This distance satisfies the following properties:

Proposition 9. If $\lambda \geq 0$, then

$$d_F^{(p)}(\lambda \tilde{E}, \lambda \tilde{F}) = |\lambda| d_F^{(p)}(\tilde{E}, \tilde{F})$$

Proposition 10.

$$d_F^{(p)}(\tilde{E} + \lambda, \tilde{F} + \lambda) = d_F^{(p)}(\tilde{E}, \tilde{F})$$

Proposition 11.

$$d_F^{(p)}(\tilde{E} + \tilde{G}, \tilde{E} + \tilde{F}) = d_F^{(p)}(\tilde{G}, \tilde{F})$$

Proof: By considering Proposition (8)

$$\begin{aligned} d_F^{(p)}(\tilde{E} + \tilde{G}, \tilde{E} + \tilde{F}) &= \left(D_F^{(p)}(\tilde{E} + \tilde{G}, \tilde{E} + \tilde{F}) \right)^{\frac{1}{p}} \\ &= \left(\sup_{\alpha \in [0, 1]} D_I^{(p)}([\tilde{E}]^\alpha + [\tilde{G}]^\alpha, [\tilde{E}]^\alpha + [\tilde{F}]^\alpha) \right)^{\frac{1}{p}} \\ &= \left(\sup_{\alpha \in [0, 1]} D_I^{(p)}([\tilde{G}]^\alpha, [\tilde{F}]^\alpha) \right)^{\frac{1}{p}} \\ &= \left(D_F^{(p)}(\tilde{G}, \tilde{F}) \right)^{\frac{1}{p}} \\ &= d_F^{(p)}(\tilde{G}, \tilde{F}) \end{aligned} \quad (16)$$

Then the proof is completed.

Proposition 12.

$$d_F^{(p)}(a, b) = |a - b|$$

(both fuzzy numbers collapse to a single point).

Proposition 13.

$$d_F^{(p)}([a, b], [c, d]) = d_I^{(p)}([a, b], [c, d])$$

(both fuzzy numbers collapse to an interval).

Definition 2. If $\tilde{A}$ and $\tilde{B}$ are two fuzzy numbers and M is a crisp number then by (13), (14)

$$d_F^{(p)}(\tilde{A}, \tilde{B}, M) = (D_F^{(p)}(\tilde{A}, \tilde{B}, M))^{\frac{1}{p}} \quad (17)$$

where

$$D_F^{(p)}(\tilde{A}, \tilde{B}, M) = \sup_{\alpha \in [0, 1]} D_I^{(p)}([A]^\alpha, [B]^\alpha, M) d\alpha \quad (18)$$

Then regarding (12)

$$\begin{aligned} D_F^{(p)}(\tilde{A}, \tilde{B}, M) &= \sup_{\alpha \in [0, 1]} \\ &\int_0^1 D_I^{(p)}([A]^\alpha, ([B_l(\alpha) - B_r(\alpha)]x + B_r(\alpha)), M) dx \end{aligned} \quad (19)$$

If $\tilde{B} = 1$ then $d_F^{(p)}(\tilde{A}, \tilde{B}, M) = d_F^{(p)}(\tilde{A}, M)$.

5. Ranking fuzzy numbers by α -distance

Ranking of fuzzy numbers has been studied by many researchers. Some researchers introduced distances and then compared fuzzy numbers with them ([2, 4, 16, 19]). Other researchers used the nearest symmetric (nonsymmetric) triangular fuzzy number or symmetric (nonsymmetric) trapezoidal fuzzy number for comparing fuzzy numbers, ([9, 13]). Each method has its own shortcoming. In this section, we propose an approach for ranking two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ based on the α -distance. For ranking two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, if $\sup(\text{supp}(\tilde{A})) < \inf(\text{supp}(\tilde{B}))$, it is evident that

$\tilde{A} < \tilde{B}$ and that the degree of $\tilde{A} < \tilde{B}$ is 1 and the degree of $\tilde{A} > \tilde{B}$ is 0; therefore, we consider two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ such that $\text{supp}(\tilde{A}) \cap \text{supp}(\tilde{B}) \neq \emptyset$. Tran and Duckstein [16] proposed a similar distance for ranking but it was not a metric distance because $d([a, b], [a, b]) \neq 0$. The proposed method in this paper for ranking two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is based on the comparison of the degree of distance between the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ and *crisp max*(M) and *crisp min*(m) where:

$$\begin{cases} M \geq \max(\text{supp}\tilde{A} \cup \text{supp}\tilde{B}) \\ \& \\ m \leq \min(\text{supp}\tilde{A} \cup \text{supp}\tilde{B}) \end{cases} \quad (20)$$

We denote the degree of distance between $\tilde{A}$ and crisp numbers $\max(M)$ and $\min(m)$ by $\gamma_d^{(p)}(\tilde{A}, M)$ and $\gamma_d^{(p)}(\tilde{A}, m)$, respectively, which are defined as follows:

$$\gamma_d^{(p)}(\tilde{A}, M) = \frac{d_F^{(p)}(\tilde{A}, M)}{d_F^{(p)}(\tilde{A}, M) + d_F^{(p)}(\tilde{A}, m)} \quad (21)$$

$$\gamma_d^{(p)}(\tilde{A}, m) = \frac{d_F^{(p)}(\tilde{A}, m)}{d_F^{(p)}(\tilde{A}, M) + d_F^{(p)}(\tilde{A}, m)} \quad (22)$$

such that $d_F^{(p)}(\tilde{A}, M)$ and $d_F^{(p)}(\tilde{A}, m)$ in (13) and (14) are the distances between the fuzzy number $\tilde{A}$ and crisp numbers $\max(M)$ and $\min(m)$, respectively.

Proposition 14.

$$\gamma_d^{(p)}(\tilde{A}, M) + \gamma_d^{(p)}(\tilde{A}, m) = 1$$

Proposition 15.

$$\gamma_d^{(p)}(\tilde{A}, M) \geq \gamma_d^{(p)}(\tilde{B}, M)$$

if only if

$$\gamma_d^{(p)}(\tilde{A}, m) \leq \gamma_d^{(p)}(\tilde{B}, m)$$

Proof: By Proposition 14, the proof is clear.

Proposition 16. If $M' > M$ and

$$\gamma_d^{(p)}(\tilde{A}, M) \geq \gamma_d^{(p)}(\tilde{B}, M)$$

then

$$\gamma_d^{(p)}(\tilde{A}, M') \geq \gamma_d^{(p)}(\tilde{B}, M)$$

Proof: The proof is clear.

Proposition 17. If $m' < m$ and

$$\gamma_d^{(p)}(\tilde{A}, m) \leq \gamma_d^{(p)}(\tilde{B}, m)$$

then

$$\gamma_d^{(p)}(\tilde{A}, m') \leq \gamma_d^{(p)}(\tilde{B}, m)$$

Proof: The proof is clear.

Definition 3. The ranking method is as follows:

(a):

$$\tilde{A} \lesssim \tilde{B} \Leftrightarrow \begin{cases} \gamma_d^{(p)}(\tilde{A}, M) \geq \gamma_d^{(p)}(\tilde{B}, M) \\ \text{or} \\ \gamma_d^{(p)}(\tilde{A}, m) \leq \gamma_d^{(p)}(\tilde{B}, m) \end{cases} \quad (23)$$

such that the degree of ranking is defined as:

$$\begin{aligned} \gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} < \tilde{B})} &= \frac{\lambda \cdot \gamma_d^{(p)}(\tilde{A}, M)}{\gamma_d^{(p)}(\tilde{A}, M) + \gamma_d^{(p)}(\tilde{B}, M)} \\ &\quad + \frac{(1-\lambda) \cdot \gamma_d^{(p)}(\tilde{B}, m)}{\gamma_d^{(p)}(\tilde{A}, m) + \gamma_d^{(p)}(\tilde{B}, m)}, \quad (24) \\ \lambda &\in [0, 1] \end{aligned}$$

λ is chosen according to the decision-maker's idea. If $\lambda = 0$, the ranking method is as presented in (22); if $\lambda = 1$, the ranking method is as provided in (21) and if $0 < \lambda < 1$, the ranking method works with *crisp min*(m) and *crisp max*(M).

(b):

$$\tilde{A} = \tilde{B} \Leftrightarrow d_F^{(p)}(\tilde{A}, \tilde{B}) = 0 \quad (25)$$

then the degree of ranking is defined as follows:

$$\gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} = \tilde{B})} = 1 \quad (26)$$

and

$$\gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} > \tilde{B})} = \gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} < \tilde{B})} = 0 \quad (27)$$

(c):

$$\tilde{A} \approx \tilde{B} \Leftrightarrow \begin{cases} \gamma_d^{(p)}(\tilde{A}, M) = \gamma_d^{(p)}(\tilde{B}, M) = \frac{1}{2} \\ \& \\ \gamma_d^{(p)}(\tilde{A}, m) = \gamma_d^{(p)}(\tilde{B}, m) = \frac{1}{2} \end{cases} \quad (28)$$

In this case, we have the maximum ambiguity for ranking, therefore the degree of ranking is defined as follows:

$$\begin{aligned} \gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} = \tilde{B})} &= \gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} > \tilde{B})} \\ &= \gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} < \tilde{B})} \quad (29) \\ &= \frac{1}{2} \end{aligned}$$

In this method, for ranking n fuzzy numbers $\tilde{A}_1, \dots, \tilde{A}_n$ we compare the degree of distance between the fuzzy numbers and any *crisp max*(M) and any *crisp min*(m) where:

$$\begin{cases} M \geq \max(\cup_{i=1}^n \text{supp}\tilde{A}_i) \\ \& \\ m \leq \min(\cup_{i=1}^n \text{supp}\tilde{A}_i) \end{cases} \quad (30)$$

If we add $\tilde{C}$ to the set, then M and m , as well as, may change, but regarding Proposition, (16)

and (17), there will be no change in the ranking of $\tilde{A}_1, \dots, \tilde{A}_n$.

Proposition 18.

$$\gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} < \tilde{B})} + \gamma_{(\lambda M + (1-\lambda)m)}^{(\tilde{A} > \tilde{B})} = 1, \quad \lambda \in [0, 1].$$

Proposition 19. If $\tilde{A} \precsim \tilde{B}$ and $\tilde{B} \precsim \tilde{C}$ then for each $\lambda \in [0, 1]$,

$$\gamma_{(\lambda.M'' + (1-\lambda)m'')}^{(\tilde{A} < \tilde{C})} \geq \max \left\{ \gamma_{(\lambda.M + (1-\lambda)m)}^{(\tilde{A} < \tilde{B})}, \gamma_{(\lambda.M' + (1-\lambda)m')}^{(\tilde{B} < \tilde{C})} \right\}$$

where

$$M \geq \max(\text{supp} \tilde{A} \cup \text{supp} \tilde{B}),$$

$$m \leq \min(\text{supp} \tilde{A} \cup \text{supp} \tilde{B}),$$

$$M' \geq \max(\text{supp} \tilde{B} \cup \text{supp} \tilde{C}),$$

$$m' \leq \min(\text{supp} \tilde{B} \cup \text{supp} \tilde{C}),$$

$$M'' \geq \max(M, M'), \quad m'' \leq \min(m, m'). \quad (31)$$

Now we consider the following reasonable properties for the ordering approaches on E (the set of fuzzy numbers)(see [17]).

A: For an arbitrary finite subset Λ of E and $\tilde{u} \in \Lambda$, $\tilde{u} \precsim \tilde{u}$.

Proof: The proof is evident by the definition of ranking.

B: For an arbitrary finite subset Λ of E and $(\tilde{u}, \tilde{v}) \in \Lambda^2$, $\tilde{u} \precsim \tilde{v}$ and $\tilde{u} \precsim \tilde{v}$, we should have $\tilde{u} \approx \tilde{v}$.

Proof: The proof is clear by the definition of ranking.

C: For an arbitrary finite subset Λ of E and $(\tilde{u}, \tilde{v}, \tilde{z}) \in \Lambda^3$, $\tilde{u} \precsim \tilde{v}$ and $\tilde{v} \precsim \tilde{z}$, we should have $\tilde{u} \precsim \tilde{z}$.

Proof: From (21), (22) and (23)

$$d_F^{(p)}(\tilde{u}, M)d_F^{(p)}(\tilde{v}, m) \geq d_F^{(p)}(\tilde{v}, M)d_F^{(p)}(\tilde{u}, m) \quad (32)$$

$$d_F^{(p)}(\tilde{v}, M')d_F^{(p)}(\tilde{z}, m') \geq d_F^{(p)}(\tilde{z}, M')d_F^{(p)}(\tilde{v}, m') \quad (33)$$

where

$$M' \geq \max(\text{supp} \tilde{v} \cup \text{supp} \tilde{z})$$

$$m' \leq \min(\text{supp} \tilde{v} \cup \text{supp} \tilde{z})$$

$$M \geq \max(\text{supp} \tilde{u} \cup \text{supp} \tilde{v})$$

$$m \leq \min(\text{supp} \tilde{u} \cup \text{supp} \tilde{v})$$

It is sufficient to show that

$$d_F^{(p)}(\tilde{u}, M'')d_F^{(p)}(\tilde{z}, m'') \geq d_F^{(p)}(\tilde{z}, M'')d_F^{(p)}(\tilde{u}, m'') \quad (34)$$

where

$$M'' \geq \max\{M, M'\}$$

and

$$m'' \leq \min\{m, m'\}$$

Therefore

$$\begin{aligned} & d_F^{(p)}(\tilde{u}, M'')d_F^{(p)}(\tilde{z}, m'') \\ & \geq d_F^{(p)}(\tilde{v}, M'')d_F^{(p)}(\tilde{z}, m'') \\ & \geq d_F^{(p)}(\tilde{v}, m'')d_F^{(p)}(\tilde{z}, M'') \\ & \geq d_F^{(p)}(\tilde{u}, m'')d_F^{(p)}(\tilde{z}, M'') \end{aligned} \quad (35)$$

D: For an arbitrary finite subset Λ of E and $(\tilde{u}, \tilde{v}) \in \Lambda^2$, $\inf \text{supp}(\tilde{u}) > \sup \text{supp}(\tilde{v})$, we should have $\tilde{u} \precsim \tilde{v}$.

Proof: The proof is obvious by the definition of ranking.

D': For an arbitrary finite subset Λ of E and $(\tilde{u}, \tilde{v}) \in \Lambda^2$, $\inf \text{supp}(\tilde{u}) > \sup \text{supp}(\tilde{v})$, we should have $\tilde{u} \succ \tilde{v}$.

Proof: The proof is clear by the definition of ranking.

E: Let Λ and Λ' be two arbitrary finite subsets of E in which $\tilde{u}$ and $\tilde{v}$ are in $\Lambda \cap \Lambda'$. We obtain the ranking order $\tilde{u} \succeq \tilde{v}$ by d_F on Λ' if and only if $\tilde{u} \succeq \tilde{v}$ by d_F on Λ .

Proof: The proof is evident by the definition of ranking.

F: Let $\tilde{u}$, $\tilde{v}$, $\tilde{u} + \tilde{z}$ and $\tilde{v} + \tilde{z}$ be elements of E. If $\tilde{u} \precsim \tilde{v}$, then $\tilde{u} + \tilde{z} \precsim \tilde{v} + \tilde{z}$.

Proof: From (21), (22) and (23)

$$d_F^{(p)}(\tilde{u}, M)d_F^{(p)}(\tilde{v}, m) \geq d_F^{(p)}(\tilde{v}, M)d_F^{(p)}(\tilde{u}, m) \quad (36)$$

where

$$M \geq \max(\text{supp} \tilde{u} \cup \text{supp} \tilde{v}),$$

$$m \leq \min(\text{supp} \tilde{u} \cup \text{supp} \tilde{v})$$

It is sufficient to show that

$$\begin{aligned} & d_F^{(p)}(\tilde{u} + \tilde{z}, M + M')d_F^{(p)}(\tilde{v} + \tilde{z}, m + m') \geq \\ & d_F^{(p)}(\tilde{v} + \tilde{z}, M + M')d_F^{(p)}(\tilde{u} + \tilde{z}, m + m') \end{aligned} \quad (37)$$

where

$$M' \geq \max(\text{supp} \tilde{z}),$$

$$m' \leq \min(\text{supp} \tilde{z})$$

It is obvious that

$$d_F^{(p)}(\tilde{u} + \tilde{z}, M + M') \geq d_F^{(p)}(\tilde{v} + \tilde{z}, M + M')$$

and

$$d_F^{(p)}(\tilde{v} + \tilde{z}, m + m') \geq d_F^{(p)}(\tilde{u} + \tilde{z}, m + m')$$

F' : Let $\tilde{u}$, $\tilde{v}$, $\tilde{u} + \tilde{z}$ and $\tilde{v} + \tilde{z}$ be elements of E . If $\tilde{u} > \tilde{v}$, then $\tilde{u} + \tilde{z} > \tilde{v} + \tilde{z}$ when $\tilde{z} \neq 0$.

Proof: If $\tilde{u} > \tilde{v}$ then

$$\max(\text{supp } \tilde{v}) < \min(\text{supp } \tilde{u})$$

It is clear that if $\tilde{z} \neq 0$ then

$$\max(\text{supp } (\tilde{v} + \tilde{z})) < \min(\text{supp } (\tilde{u} + \tilde{z}))$$

hence,

$$\tilde{u} + \tilde{z} > \tilde{v} + \tilde{z}$$

G: Let $\tilde{u}$, $\tilde{v}$, $\tilde{u}\tilde{z}$ and $\tilde{v}\tilde{z}$ be elements of E and

$$\tilde{z} \geq 0(\min(\text{supp } \tilde{z}) \geq 0)$$

$\tilde{u} \preceq \tilde{v}$, implies $\tilde{u}\tilde{z} \preceq \tilde{v}\tilde{z}$.

Proof: It is sufficient to show that

$$\begin{aligned} d_F^{(p)}(\tilde{u}\tilde{z}, M' M) d_F^{(p)}(\tilde{v}\tilde{z}, m' m) &\geq \\ d_F^{(p)}(\tilde{v}\tilde{z}, M' M) d_F^{(p)}(\tilde{u}\tilde{z}, m' m) &\end{aligned} \quad (38)$$

where

$$M' \geq \max(\text{supp } \tilde{z})$$

$$m' \leq \min(\text{supp } \tilde{z})$$

$$M \geq \max(\text{supp } \tilde{u} \cup \text{supp } \tilde{v})$$

$$m \leq \min(\text{supp } \tilde{u} \cup \text{supp } \tilde{v})$$

It is sufficient to show that

$$\begin{cases} d_F^{(p)}(\tilde{u}\tilde{z}, M' M, s) \geq d_F^{(p)}(\tilde{v}\tilde{z}, M' M) \\ d_F^{(p)}(\tilde{v}\tilde{z}, m' m) \geq d_F^{(p)}(\tilde{u}\tilde{z}, m' m) \end{cases} \quad (39)$$

We demonstrate that the proofs of $d_F^{(p)}(\tilde{u}\tilde{z}, M' M) \geq d_F^{(p)}(\tilde{v}\tilde{z}, M' M)$ and that of $d_F^{(p)}(\tilde{v}\tilde{z}, m' m) \geq d_F^{(p)}(\tilde{u}\tilde{z}, m' m)$ are similar. By (19)

$$D_F^{(p)}(\tilde{u}\tilde{z}, M' M) = \sup_{\alpha \in [0,1]}$$

$$\int_0^1 D_I^{(p)}([u]^\alpha \cdot ((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha)), M' M) dx \quad (40)$$

Considering (13), it is sufficient to show that

$$D_F^{(p)}(\tilde{u}\tilde{z}, M' M) \geq D_F^{(p)}(\tilde{v}\tilde{z}, M' M)$$

It is clear that

$$\frac{D_F^{(p)}(\tilde{u}\tilde{z}, M' M)}{D_F^{(p)}(\tilde{v}\tilde{z}, M' M)} \geq 1$$

if

$$\frac{\sup_{\alpha \in [0,1]} D_I^{(p)}([u]^\alpha \cdot ((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha)), M' M)}{\sup_{\alpha \in [0,1]} D_I^{(p)}([v]^\alpha \cdot ((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha)), M' M)} \geq 1$$

However,

$$\frac{\sup_{\alpha \in [0,1]} D_I^{(p)}([u]^\alpha \cdot ((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha)), M' M)}{\sup_{\alpha \in [0,1]} D_I^{(p)}([v]^\alpha \cdot ((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha)), M' M)}$$

$$= \frac{\sup_{\alpha \in [0,1]} D_I^{(p)}([u]^\alpha \cdot \frac{((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha))}{M'}, M)}{\sup_{\alpha \in [0,1]} D_I^{(p)}([v]^\alpha \cdot \frac{((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha))}{M'}, M)}$$

If

$$k(\alpha, x) = \frac{((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha))}{M'}$$

then

$$0 < k(\alpha, x) < 1$$

hence

$$\frac{\sup_{\alpha \in [0,1]} D_I^{(p)}([u]^\alpha \cdot \frac{((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha))}{M'}, M)}{\sup_{\alpha \in [0,1]} D_I^{(p)}([v]^\alpha \cdot \frac{((z_l(\alpha) - z_r(\alpha))x + z_r(\alpha))}{M'}, M)} =$$

$$\frac{\sup_{\alpha \in [0,1]} D_I^{(p)}([u]^\alpha \cdot k(\alpha, x), M)}{\sup_{\alpha \in [0,1]} D_I^{(p)}([v]^\alpha \cdot k(\alpha, x), M)} =$$

$$\frac{\sup_{\alpha \in [0,1]} k(\alpha, x) D_I^{(p)}([u]^\alpha, \frac{M}{k(\alpha, x)})}{\sup_{\alpha \in [0,1]} k(\alpha, x) D_I^{(p)}([v]^\alpha, \frac{M}{k(\alpha, x)})} \geq$$

$$\frac{\sup_{\alpha \in [0,1]} k(\alpha, x) D_I^{(p)}([u]^\alpha, T)}{\sup_{\alpha \in [0,1]} k(\alpha, x) D_I^{(p)}([v]^\alpha, T)} = 1$$

Because, regarding the assumption, $\tilde{u} \preceq \tilde{v}$ implies

$$\frac{\sup_{\alpha \in [0,1]} D_I^{(p)}([u]^\alpha, M)}{\sup_{\alpha \in [0,1]} D_I^{(p)}([v]^\alpha, M)} \geq 1 \quad (41)$$

with $M \geq \max(\text{supp } \tilde{u} \cup \text{supp } \tilde{v})$. Therefore

$$\frac{D_F^{(p)}(\tilde{u}\tilde{z}, M' M)}{D_F^{(p)}(\tilde{v}\tilde{z}, M' M)} \geq 1 \quad (42)$$

or

$$D_F^{(p)}(\tilde{u}\tilde{z}, M' M) \geq D_F^{(p)}(\tilde{v}\tilde{z}, M' M) \quad (43)$$

Remark 1. If $\tilde{u} \preceq \tilde{v}$ then $-\tilde{u} \succeq -\tilde{v}$.

6. Numerical example

The results of comparing our proposed method with some other ranking methods ([16]) are summed up in Figure 1 and Table 1.

Figure 1: Figures of ranking.

Regarding to Figure 1 and Table 1 we have:

(a): $A_1 = (.4, .9, 1)$, $A_2 = (.4, .7, 1)$, $A_3 = (.4, .5, 1)$ then

$$A_3 \preceq A_2 \preceq A_1$$

(b): $A_1 = (.2, .5, .8)$, $A_2 = (.4, .5, .6)$ then

$$A_1 \approx A_2$$

Figure 1: $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$

Figure 2: $(\tilde{A}_1, \tilde{A}_2)$

Figure 3: $((\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$

Figure 4: $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$

(c): $A_1 = (.5, .7, .9), A_2 = (.3, .7, .9), A_3 = (.3, .4, .7, .9)$ then

$$A_3 \lesssim A_2 \lesssim A_1$$

(d): $A_1 = (.3, .5, .8, .9), A_2 = (.3, .5, .9), A_3 = (.3, .5, .7)$ then

$$A_3 \lesssim A_2 \lesssim A_1$$

(e): $A_1 = (.3, .3, 1), A_2 = (.1, .8, .8)$ then

$$A_1 \lesssim A_2$$

Figure 5: $(\tilde{A}_1, \tilde{A}_2)$

7. Conclusions

In this paper, a metric distance on interval numbers was introduced using the α -cut concept. Subsequently, it was extended to fuzzy numbers. Comparison of fuzzy numbers was made based on the degree of distance between the fuzzy numbers and *crisp max*(M) and *crisp min*(m). If we add a fuzzy number to the set of fuzzy numbers, then M and m , as well as, may change, but there will be no changes in the ranking, which is the very useful property in the industrial problems. By some numerical examples, it was shown that the proposed method is more consistent with institution than the previous ranking methods.

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Table 1: Comparison of fuzzy numbers by some methods.

Methods		A_1	A_2	A_3	A_1	A_2
Yager	F_1	0.760	0.700	0.630	0.500	0.500
	F_2	0.900	0.760	0.660	0.610	0.540
	F_3	0.800	0.700	0.600	0.600	0.500
Ba – Kw		1.000	0.740	0.600	1.000	1.000
	Bald – Gui	1 : P	0.420	0.330	0.330	0.270
	g	0.550	0.400	0.340	0.300	0.240
	$r : a$	0.280	0.230	0.220	0.200	0.230
Kerre		1.000	0.860	0.760	0.910	0.910
Jain	$k = 1$	0.900	0.760	0.660	0.730	0.670
	$k = 2$	0.840	0.650	0.540	0.600	0.480
	$k = 1/2$	0.950	0.860	0.780	0.830	0.800
Dub – Pra	PD	1.000	0.740	0.600	1.000	1.000
	PSD	0.740	0.230	0.160	0.730	0.240
	ND	0.630	0.380	0.180	0.270	0.760
Lee – Li	NSD	0.260	0.000	0.000	0.000	0.000
	U.m	0.760	0.700	0.630	0.500	0.500
	U.G	...	...	...	0.120	0.040
	P.m	0.800	0.700	0.600	0.500	0.500
	P.G	...	...	...	0.090	0.030
For – Rou	F_0	0.800	0.700	0.600	0.500	0.500
Tran – Duc	$D_{max}; f = x$	0.187	0.308	0.442	0.505	0.501
	$D_{min}; f = x$	0.838	0.704	0.573	0.505	0.501
	$D_{max}; f = 1$	0.231	0.316	0.416	0.510	0.501
	$D_{min}; f = 1$	0.808	0.707	0.611	0.510	0.501
d_F	$\gamma_d^{(2)}(., M);$	0.409	0.500	0.591	0.500	0.500
d_F	$\gamma_d^{(2)}(., m);$	0.591	0.500	0.409	0.500	0.500

Table 1.(continued)

	c			d		e	
A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2
0.700	0.630	0.570	0.620	0.560	0.500	0.610	0.530
0.750	0.750	0.750	0.810	0.640	0.580	0.660	0.690
0.700	0.650	0.570	0.620	0.540	0.500	0.580	0.560
1.000	1.000	1.000	1.000	1.000	1.000	0.840	1.000
0.370	0.270	0.270	0.450	0.370	0.270	0.420	0.330
0.420	0.350	0.350	0.350	0.400	0.280	0.440	0.370
0.270	0.190	0.190	0.310	0.280	0.210	0.340	0.240
1.000	0.910	0.750	1.000	0.850	0.750	0.960	0.890
0.820	0.820	0.820	0.900	0.690	0.640	0.660	0.690
0.710	0.710	0.710	0.820	0.560	0.450	0.530	0.510
0.890	0.890	0.890	0.940	0.800	0.770	0.780	0.810
1.000	1.000	1.000	1.000	1.000	1.000	0.840	1.000
0.500	0.500	0.500	0.800	0.200	0.000	0.540	0.460
0.670	0.350	0.000	0.500	0.500	0.500	0.540	0.460
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.160
0.700	0.630	0.570	0.620	0.560	0.500	0.610	0.530
...	...	...	...	...	...	...	...
0.700	0.650	0.580	0.630	0.550	0.500	0.530	0.580
...	...	...	...	...	...	...	...
0.700	0.650	0.575	0.625	0.550	0.500	0.490	0.610
0.304	0.342	0.457	0.395	0.473	0.502	0.574	0.355
0.702	0.671	0.585	0.650	0.539	0.502	0.451	0.673
0.307	0.365	0.445	0.398	0.462	0.504	0.531	0.417
0.703	0.658	0.590	0.639	0.560	0.504	0.512	0.628
0.357	0.464	0.510	0.522	0.536	0.643	0.544	0.453
0.643	0.536	0.490	0.478	0.464	0.357	0.456	0.547