assume that the ordering is: (⟨4, Very High⟩, ⟨8, High⟩, ⟨10, Medium⟩, ⟨6, Low⟩, ⟨2, Very Low⟩).

Note that, in this example, we associate each linguistic label with an order inducing value. However, in the usual IOWA, the order inducing variables are associated with the considered arguments, such as an order induced by the importance of each argument or its consistency [7]. When using linguistic variables, our suggestion seems to be very natural because the linguistic values have a semantic interpretation for the decision maker.

Moreover, this model regarding the use of order inducing variables with linguistic labels can be extended when using numerical values. In this case, the idea is the same but now we have numbers. That is, situations when, for example, in a numerical scale from 1 to 10 we get that the optimal ranking is not in descending or ascending order. Instead, we have for example: 7 > 6 > 8 > 5 > 9 > 4 > 10 > 3 > 2 > 1. Note that here ">" means "preferred to". Obviously, in this case we cannot use the OWA aggregation and we need to use the IOWA aggregation.

In addition to the possibility that the decision maker defines his personal policy for inducing the ordering, a further interesting issue arises when dealing with unbalanced sets of terms. As it has been said in the introduction, unbalanced terms permit to define linguistic variables with different granularity and distribution of terms on the positive and on the negative terms. This gives the user a vocabulary with different types of uncertainty on the evaluations. For example, in Figure 1, label *L* has a larger support that label *VH* (i.e. the set of values of x with $\mu_{s_i}(x) \geq 0$ is larger).

Consequently, we have labels with different degrees of uncertainty. This uncertainty should be taken into account during the aggregation process, as each label is providing a different amount of information about the evaluated alternative.

In fact, if we consider that both triangular and trapezoidal fuzzy sets can be associated to the labels (as in Figure 1), then, the uncertainty of labels is not only related to their support intervals in the reference domain but also on their kernel (i.e. the set of points of value 1).

Having into account these different features of the definition of the linguistic variable, we propose to use some measure of uncertainty for the linguistic labels as the order inducing criterion of the aggregation. So, the arguments will be ordered by decreasing uncertainty. In this way, the contribution of precise labels is prioritized while the effect of uncertain labels is reduced.

In the literature [16, 20-22], two types of uncertainty are recognized: (1) *specificity*, related to the measurement of imprecision, which is based on the cardinality of the set, and (2) *fuzziness*, or entropy, which measures the vagueness of the set as a result of having imprecise boundaries.

Definition 2. Measure of Specificity [22]:
Let X be a set and let $[0,1]^X$ be the class of fuzzy sets on X. A measure of specificity is a function *Sp*: $[0,1]^X \rightarrow [0,1]$ such that:

1. $Sp(\emptyset)=0$
2. $Sp(\mu)=1$ if and only if μ is a singleton
3. If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $Sp(\mu) \geq Sp(\eta)$.

In [20], the following specificity measure, for a fuzzy set *A* defined on *X*, is defined as a generalization of other previous formulations:

$$Sp(A) = T\left(\alpha_{\text{sup}}, N\left(\int_0^{\alpha_{\text{sup}}} M(A_\alpha)d\alpha\right)\right) \qquad (6)$$

being *T* a T-norm, *N* a negation operator and *M* a fuzzy measure.

In Eq. 7 a special case of Eq. 6 is given, by considering the T-norm *min*, the standard negation N=1-x and the Lebesgue-Stieltjes fuzzy measure $M([a,b]) = b - a$. Taking a normalized fuzzy set (with $\alpha_{\text{sup}} = 1$), and with these parameters, the specificity can be calculated as:

$$Sp(A) = 1 - \frac{\text{area under } A}{b-a} \qquad (7)$$

Definition 3. Measure of Fuzziness [23]:
Let X be a set and let $[0,1]^X$ be the class of fuzzy sets on X. A measure of fuzziness is a function Fz: $[0,1]^X \rightarrow [0,1]$ such that:

1. $Fz(A)=0$ iff *A* is a crisp set
2. $Fz(A)=1$ iff $\forall x \in X, A(x) = 1/2$
3. $Fz(A) \leq Fz(B)$ if *A* is less fuzzy than *B*, i.e. $A(x) \leq B(x) \leq 1/2$ or $A(x) \geq B(x) \geq 1/2$ for every $x \in X$.

The most common way to calculate the fuzziness is in terms of the lack of distinction between the fuzzy set *A* and its complement A^C. A general definition of this type of fuzziness measures is based on an aggregation operator *h* and a distance function *d*, as:

$$Fz(A) = h_{x \in A}(d(A(x), A^C(x))) \qquad (8)$$

For the case of continuous domains, and taking the standard negation operation and the Hamming distance, Eq. 8 corresponds to:

$$Fz(A) = 1 - \frac{1}{b-a}\int_a^b |2 \cdot A(x) - 1| \qquad (9)$$

Specificity and fuzziness refer to two different characteristics of fuzzy sets. *Specificity* (or its counterpart *non-specificity* [24]) measures the degree of truth of the sentence "containing just one element". Fuzziness is measuring the difference from a crisp set. For decision making purposes, it seems desirable to have labels that correspond to single elements, rather than to large sets of values, which may difficult the selection of the appropriate alternative. For this reason, we propose to use a measure of specificity as order inducing variable in the aggregation of linguistic terms that are qualifying a set of alternatives in a decision making process.

In case of having ties between different terms with the same specificity, a second ordering criterion may be the fuzziness associated to the set. An increasing ordering of fuzziness will be considered, as we prefer the terms with less uncertainty. If this second criterion also leads to some ties, a decreasing ordering on the preference scale associated to the terms can be considered (decreasing order of the linguistic labels in *S*). In Fig. 2, we show two fuzzy sets with the same specificity (*Sp*(*A*)=*Sp*(*B*)=0.9) according to Eq. 7 but different fuzziness (Eq. 9) (*Fz*(*A*)=0.1 and *Fz*(*B*)=0.05). In this example, the set *A* is more fuzzy than *B*, so *B* is preferred.

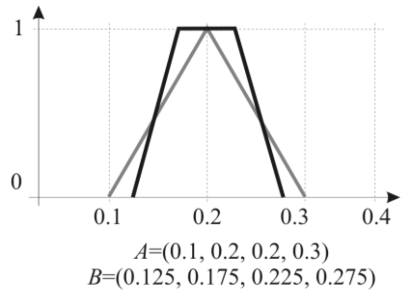

Fig. 2: Two fuzzy sets with the same specificity and different fuzziness

According to this procedure, the IULOWA operator applied to a set or arguments $\{a_1,\ldots,a_m\}$ we calculate the induced aggregation according to the uncertainty of those *m* terms using IULOWA (5) where *B* is the induced ordering vector, such that $B=(b_1,b_2,\ldots,b_m)$ satisfies these conditions:

- $\forall k \;\; 1 \le k < m \;\; Sp(b_k) \ge Sp(b_{k+1})$
- $\forall k \;\; 1 \le k < m$ if $Sp(b_k) = Sp(b_{k+1})$ then $Fz(b_k) \le Fz(b_{k+1})$
- $\forall k \;\; 1 \le k < m$ if $Sp(b_k) = Sp(b_{k+1})$ and $Fz(b_k) = Fz(b_{k+1})$ then $b_k > b_{k+1}$ according to the linguistic scale *S*

Notice that if the set of terms corresponds to crisp numbers, IULOWA is reduced to the OWA operator.

OWA weights w_i are used to define different conjunction/disjunction aggregation models [19]. As proposed by [3, 6], the procedure of inclusion of an additional variable into the OWA may involve also the transformation of the set of weights.

We also propose to modify the set of weights associated to the arguments by taking into consideration the specificity of the values that are aggregated. In this way, the more specific values should have a higher weight, whereas the less specific terms (less reliable) should have a lower weight.

Considering the family of fuzzy quantifiers proposed by Yager [4], the set of weights is obtained by the expression:

$$w_k = Q\left(\frac{S(k)}{S(n)}\right) - Q\left(\frac{S(k-1)}{S(n)}\right), \qquad (11)$$

where $S(k)=\sum_{i=1}^{k} u_{\sigma(i)}$ and σ is the permutation according to the order inducing procedure established before (Eq. 10).

The properties of the quantifier function must be taken into account in order to generate a coherent set of weights for the OWA operator. Considering the usual quantifier $Q(r)=r^a$ [4], when $a\in[0,1]$ then the weighting function is concave, satisfying that the largest the specificity, the higher the weight w_k of the corresponding argument [6]. It is worth to note that with $a\in[0,1]$ the aggregation policy is disjunctive, facilitating the replaceability of uncertain evaluations by the most specific (and less fuzzy) available values.

5. Examples

In this section three examples are analysed. The first is an example to show how the sorting of terms is performed according to the measures of specificity and fuzziness. The second example studies how the specificity can influence the generation of the aggregation weights. Finally, the third example is devoted to use the IULOWA operator in a decision problem in which research papers have to be evaluated.

5.1. Order induction example

As previously stated, the order of aggregation of the linguistic fuzzy terms is found by sorting the labels we want to aggregate according to their specificity. Ties between terms with the same measure of specificity are solved by taking into account the fuzziness of each fuzzy set. If that measure still has any ties, terms will finally be ordered according to the index (i.e. position) of the terms in the linguistic scale *S*, (assuming that the highest positions are the best).

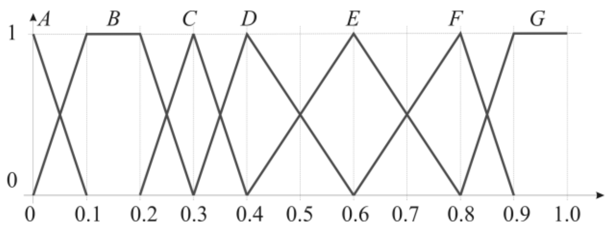

Fig. 3: Linguistic variable with 7 terms (test 1)

In this example we will show how would be sorted the terms depicted in Fig. 3. The information regarding each of the terms necessary to conduct that sorting procedure is indicated in Table 1. Specificity is calculated following Eq. 7, whereas fuzziness is obtained using Eq. 9.

Term	*Definition*	*Index*	*Specificity*	*Fuzziness*
A	(0.0,0.0,0.0,0.1)	0	0.95	0.05
B	(0.0,0.1,0.2,0.3)	1	0.8	0.10
C	(0.2,0.3,0.3,0.4)	2	0.9	0.10
D	(0.3,0.4,0.4,0.6)	3	0.85	0.15
E	(0.4,0.6,0.6,0.8)	4	0.8	0.20
F	(0.6,0.8,0.8,0.9)	5	0.85	0.15
G	(0.8,0.9,1.0,1.0)	6	0.85	0.05

Table 1: Uncertainty measures for the terms in test 1

Taking into account the specificity, the labels are ordered as *A*>*C*>(*D*,*F*,*G*)>(*B*,*E*). Note that there are two ties: the first one between *D*, *F* and *G* (with *Sp*=0.85), and the second one between *B* and *E* (*Sp*=0.8). Using the fuzziness measure to solve the ties, we put *G*(Fz*=0.05*) before *D* and *F* (*Fz*=0.15) in the first tie, while *B* (*Fz*=0.10) goes before *E* (*Fz*=0.20) in the second tie. As we can see, the measure of fuzziness is still unable to decide the order between *D* and *F*, so we use the index of the terms to decide the position, putting *F* (index=5) before *D* (index=3). Thus, the induced order according to the procedure proposed in this paper (Eq. 10) is: *A*>*C*>*G*>*F*>*D*>*B*>*E*.

5.2. Weight generation example

Weight generation is conducted by using fuzzy quantifiers and taking into account the measure of specificity of each term we want to aggregate. We have tested how the specificity of the terms modifies the weights that are finally associated to each argument.

We will use the fuzzy quantifier $Q(r) = r^a$ to generate the weighting policy for the OWA operator. Table 2 shows the weights obtained without taking into account the specificities. Tests have been done considering several values of the parameter *a*, ranging from 0.1 (where we mostly base the result on the first argument) to 1 (which corresponds to an arithmetic average of the arguments, as the weights are equal for all the values).

a[*]	*Weights*
0.1	(0.851, 0.061, 0.038, 0.028, 0.022)
0.25	(0.668, 0.127, 0.085, 0.066, 0.054)
0.5	(0.447, 0.185, 0.142, 0.120, 0.106)
0.75	(0.299, 0.204, 0.179, 0.164, 0.154)
1	(0.200, 0.200, 0.200, 0.200, 0.200)

[*] Values in the quantifier function

Table 2: Weights obtained without specificity

Two tests have been done. The first one is based on the linguistic variable with 7 terms represented in Fig. 3. We consider the generation of weights for the values (*A*,*C*,*F*,*B*,*B*) with specificities (0.95,0.9,0.85,0.8,0.8) respectively. The results are shown in Table 3.

a	*Weights*
0.1	(0.860, 0.059, 0.036, 0.025, 0.020)
0.25	(0.686, 0.124, 0.080, 0.060, 0.050)
0.5	(0.470, 0.186, 0.136, 0.110, 0.098)
0.75	(0.322, 0.209, 0.174, 0.152, 0.143)
1	(0.221, 0.209, 0.198, 0.186, 0.186)

Table 3: Weights obtained in test 1

In this test, the specificities of the terms that are aggregated are very similar among them. For this reason, the weights in Table 3 are quite similar to the ones obtained in Table 2 when specificity was not considered. This illustrates that when the specificity (i.e. confidence) on the terms is similar, the weights are almost not modified.

For the second test we have used another set of terms with different degrees of specificity, as shown in Fig. 4. In this case, we aggregate the values (*E*, *B*, *B*, *C*, *C*) with specificities (0.95, 0.8, 0.8, 0.5, 0.5) respectively.

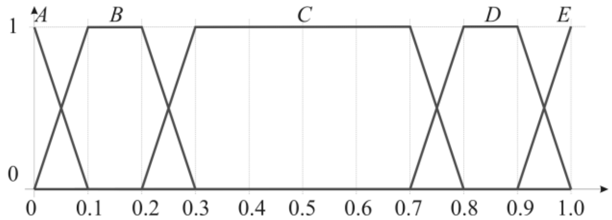

Fig. 4: Linguistic variable with 5 terms (test 2)

a	*Weights*
0.1	(0.876, 0.056, 0.036, 0.017, 0.015)
0.25	(0.719, 0.119, 0.083, 0.042, 0.037)
0.5	(0.517, 0.187, 0.145, 0.079, 0.072)
0.75	(0.372, 0.216, 0.192, 0.112, 0.108)
1	(0.268, 0.225, 0.225, 0.141, 0.141)

Table 4: Weights obtained in test 2

In this second test the two last terms have a low specificity (0.5) considering the rest of values for the first three terms (0.95 to 0.8). The results given in Table 4 show that this difference affects the weights as expected, giving more weight to the less uncertain terms. We can see a notable increase of the global weight of the first three terms and a decrease of the weight of the last two terms.

5.3. A complete example of IULOWA

Let us consider that the terms in Fig. 4 represent the evaluations the reviewers can give when reviewing a research paper, being A="Disaster", B="Bad", C="Normal", D="Good", E="Excellent". We consider that the reviewers who give the most precise evaluations (such as "Excellent" or "Disaster") are the ones with greatest knowledge in the paper's content or, at least, are more convinced to decide that extreme score. This is why we want to increase the weight we give to those specific evaluations rather than the ones that just give a "Normal" rating, which is quite unspecific. Thus, we are using an order inducing process because the information is reordered according to an additional criterion different from the values of the arguments.

In this example, we have five evaluations to aggregate in order to decide a global score for a paper, using the IULOWA operator. Those evaluations are (C, C, D, A, C).

The first step is to calculate the order in which the evaluations will be aggregated according to the measures of uncertainty given in this paper (Eq. 7 and

Eq. 9). Those measures are given in Table 5. The order of the arguments is, then, (A, D, C, C, C).

Term	*Definition*	*Index*	*Specificity*	*Fuzziness*
A	(0.0,0.0,0.0,0.1)	0	0.95	0.05
B	(0.0,0.1,0.2,0.3)	1	0.8	0.10
C	(0.2,0.3,0.7,0.8)	2	0.5	0.10
D	(0.7,0.8,0.9,1.0)	3	0.8	0.10
E	(0.9,1.0,1.0,1.0)	4	0.95	0.05

Table 5: Term set uncertainty measures

After sorting the arguments, we have used the *Sp* measure (Eq. 7) to generate the weighting vector as done in the previous example (Eq. 11), using $Q(r) = r^a$ with a= 0.75. The resulting weighting vector is W= (0.398, 0.231, 0.130, 0.123, 0.118).

The IULOWA aggregation begins following (Eq. 5):

$$IULOWA_W\ (A, D, C, C, C) = W \cdot B^T =$$
$$= C^5\{w_k, b_k, k = 1,..,5\} =$$
$$= 0.398 \otimes A \oplus (1 - 0.398) \otimes C^4\{\beta_h, b_h, h = 2,..,5\}$$

Now, we have to evaluate C^4 as:

$$C^4\{\beta_h, b_h, h = 2,..,5\} =$$
$$\left(\left(\frac{0.231}{0.602}\right) \otimes D \oplus \left(1 - \frac{0.231}{0.602}\right) \otimes C^3\{\beta_h, b_h, h = 3,..,5\}\right)$$

, where

$$C^3\{\beta_h, b_h, h = 3,..,5\} =$$
$$\left(\left(\frac{0.130}{0.371}\right) \otimes C \oplus \left(1 - \frac{0.130}{0.371}\right) \otimes C^2\{\beta_h, b_h, h = 4,..,5\}\right)$$

In the last step, C^2 is calculated as follows:

$$C^2\{\beta_h, b_h, h = 4,..,5\} =$$
$$\left(\left(\frac{0.123}{0.241}\right) \otimes C \oplus \left(1 - \frac{0.123}{0.241}\right) \otimes C\right) = s_k$$

Now, the two terms to aggregate are s_j=*C* and s_i=*C*. In this case it is trivial to find the aggregation result: *C*. Then, we continue with the same approach. The two terms to aggregate to obtain C^3 are the same: s_j=*C* and s_i=*C*, obtaining again a *C* score.

The next terms to aggregate, to calculate C^4, are s_j=*D* and s_i=*C*. Following the ULOWA convex combination (Eq. 3), we find an intermediate point between these two terms, which is calculated taking into account the centers of gravity of the corresponding fuzzy sets as well as their weights. This point is $\delta = (0.634, 0.634, 0.634, 0.634)$. It is necessary then to calculate the similarity of that intermediate point with all the terms comprised between *C* and *D*. The similarities of *C* and *D* with δ are 0.78 and 0.744 respectively, so the aggregation result of *C* and *D* is *C*.

The final step is to calculate C^5 by aggregating *A* and *C*. The intermediate point between *A* and *C* is in this case $\delta = (0.398, 0.398, 0.398, 0.398)$. The similarities of *A*, *B* (which is between *A* and *C*) and *C* with δ are 0.626, 0.748 and 0.736 respectively, so the aggregation result of *A* and *C*, and the final aggregation result, is *B*.

Note that if the same process is done without taking into account the specificities when sorting the labels and calculating the weights (as in ULOWA), the aggregation order would have been (D, C, C, C, A) and the weighting vector (0.299, 0.204, 0.179, 0.164, 0.154). In this case we obtain that the final aggregation result is *C*. This is not adequate if we want to increase the importance of the most precise evaluations, *A* in this case.

6. Conclusions and future work

The use of the IULOWA operator permits to deal with complex reordering processes by using order inducing variables. Thus, we are able to deal with problems where the highest results may not be the optimal ones, because there are other criteria that influence the ordering of the arguments. In particular, the IOWA operator defined in [5] has been extended to the case of dealing with unbalanced linguistic variables that use unbalanced fuzzy sets. Unbalanced sets of terms introduce the property of dealing with different degrees of uncertainty in the values that are aggregated.

The paper proposed a procedure to use the measurement of uncertainty as order inducing criterion in IULOWA. With this approach, the decision is based on the less uncertain values, which have more confidence from the user point of view. The concept of minimum uncertainty is interpreted as having the maximum specificity and the minimum fuzziness, two well-known measures in fuzzy theory. Ties are solved by taking as preference degree the scale of evaluation.

It can be easily seen that we have defined a more general framework that includes the ULOWA operator when all the terms have the same specificity and fuzziness. It is also reduced to the LOWA operator if the terms are balanced. In fact, the IULOWA operator provides a wide range of families of unbalanced linguistic aggregation operators following the methodology used in the OWA literature [1, 16, 18]. For example, we have the unbalanced linguistic minimum, the unbalanced linguistic maximum, the unbalanced linguistic average, the unbalanced linguistic weighted average or the unbalanced linguistic median.

The paper also shows that it is interesting to modify the weighting policy according to the level of uncertainty to make a coherent aggregation of the values.

In future research, we expect to develop further aggregation operators for unbalanced linguistic variables, for example, including the possibility of introducing the importance of the variables. We will also study the applicability of this approach in several areas such as decision making theory or online recommender systems.

Acknowledgements

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