

Finite-Time Synchronization of Chaos Gyros via Terminal Sliding Mode Control

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Abstract—The problem of finite-time synchronization between two chaotic gyros with uncertain and disturbances is investigated. On the basis of a double power reaching law, a nonsingular terminal sliding mode control algorithm was proposed to restrain chattering and improve convergence speed of terminal mode control. First, a new nonsingular terminal sliding surface is introduced and its finite-time convergence to the equilibrium is proved. Then, a sliding mode controller is proposed based on a double power reaching law to force the trajectories of the synchronization error system onto the sliding surface and remain on it forever, and the finite-time synchronization conditions are obtained. Finally, simulation results are presented to illustrate the effectiveness of the design.

Keywords—chaos gyros; finite-time synchronization; nonsingular terminal sliding mode; double power reaching law

I INTRODUCTION

Synchronization of the chaotic dynamical systems has gained a great deal of interest among researchers and engineers from variety of research fields in recent years [1-5]. In this line, many different methods have been applied theoretically and experimentally to synchronize chaotic systems, such as active control, adaptive control, linear control and sliding mode control [6-10].

The gyro is one of the most interesting dynamical systems. Gyros have found useful applications in optics, navigation, aeronautics and space engineering fields. The pioneering paper on the concept of chaotic motion in gyros was not presented until 1981. Chen [11] analyzed the dynamics of a symmetric gyro with linear plus-cubic damping subjected to a harmonic excitation. Lei [12] proposed the active control method to achieve complete synchronization between two identical chaotic gyros. Yan [13] addressed the problem of the chaotic gyros with fully unknown parameters using adaptive sliding mode control. Yau [14] has developed fuzzy controllers for synchronizing two uncertain chaotic gyros. So

far, most of the existing results related to synchronization mainly focused on asymptotic synchronization. From a practical point of view, however, it is more valuable that the synchronization objective is realized in a finite time. Furthermore, the finite-time synchronization has demonstrated better robustness and disturbance rejection properties.

In this paper, we present a design scheme for the terminal sliding mode (TSM) control which can realize chaotic synchronization between two chaotic gyros with uncertain and disturbances in finite time.

II SYSTEM DESCRIPTION

The dynamics of a symmetrical gyro with linear-plus-cubic damping of the angle θ can be expressed as [11]:

$$\ddot{\theta} + c_1 \dot{\theta} + c_2 \dot{\theta}^3 + \alpha^2 \frac{(1 - \cos \theta)^2}{\sin^3 \theta} - \beta \sin \theta = f \sin \omega t \sin \theta \quad (1)$$

Where $f \sin \omega t$ is a parametric excitation that models the base excitation, $c_1 \dot{\theta}$ and $c_2 \dot{\theta}^3$ are the linear and the nonlinear damping terms, respectively, and the term $\alpha^2 (1 - \cos \theta)^2 / \sin^3 \theta - \beta \sin \theta$ is the nonlinear resilience. Given the states $x_1 = \theta$, $x_2 = \dot{\theta}$, this system can be transformed into the following nominal form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -c_1 x_2 - c_2 x_2^3 - \alpha^2 \frac{(1 - \cos x_1)^2}{\sin^3 x_1} + (\beta + f \sin \omega t) \sin x_1 \end{cases} \quad (2)$$

The nonlinear gyro system exhibits chaotic behavior for the specific parameter values of $\alpha^2 = 100$, $\beta = 1$, $c_1 = 0.5$, $c_2 = 0.05$, $\omega = 2$, $f = 35.5$. The chaotic

motion and the strange attractor of system (2) with initial conditions of $x_1(0) = 1, x_2(0) = -1$ are illustrated in Fig. 1.

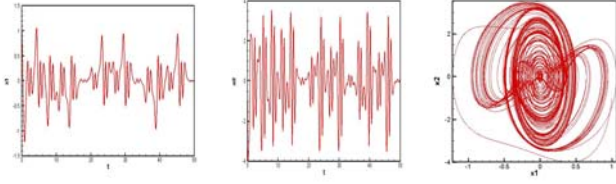


FIGURE 1. STATE TRAJECTORIES OF SYSTEM AND THE CHAOTIC ATTRACTOR OF SYSTEM (1).

Assume that two above-mentioned systems are given, one is the drive system with the subscript 3, and the other is the response system with the subscript 4. The drive system is given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -c_1 x_2 - c_2 x_2^3 - \alpha^2 \frac{(1 - \cos x_1)^2}{\sin^3 x_1} + (\beta + f \sin \omega t) \sin x_1 \end{cases} \quad (3)$$

The response system is given by

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -c_1 y_2 - c_2 y_2^3 - \alpha^2 \frac{(1 - \cos y_1)^2}{\sin^3 y_1} + (\beta + f \sin \omega t) \sin y_1 \\ \quad + \Delta f(y_1, y_2) + d(t) + u(t) \end{cases} \quad (4)$$

where $x(t) = (x_1, x_2)^T$ is the state vector of the drive gyros, $y(t) = (y_1, y_2)^T$ is the state vector of the response system, $\Delta f(y_1, y_2)$ and $d(t)$ are unknown model uncertainties and external disturbances of the response system, respectively; and $u(t)$ is the control input to be designed later. To solve the finite-time synchronization problem, the synchronization error between the drive (3) and response systems (4) can be defined as $e(t) = (e_1, e_2)^T$. Let the error variables be $e_1 = x_2 - y_2, e_2 = y_2 - y_1$. Therefore, with subtracting Eq. (4) from Eq. (3), the error dynamics is obtained as follows:

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = -c_1 e_2 - \alpha^2 \phi_1(x_1, y_1) - c_2 \phi_2(x_2, y_2) \\ \quad + (\beta + f \sin \omega t) \phi_3(x_1, y_1) + \Delta f(y_1, y_2) + d(t) + u(t) \end{cases} \quad (5)$$

Where $\phi_1(x_1, y_1) = (1 - \cos y_1)^2 / \sin^3 y_1 - (1 - \cos x_1)^2 / \sin^3 x_1$, $\phi_2(x_2, y_2) = y_2^3 - x_2^3$; $\phi_3(x_1, y_1) = \sin y_1 - \sin x_1$.

Assumption 1 It is assumed that the external $\Delta f(y_1, y_2)$ and disturbances $d(t)$ are norm-bounded:

$$|\Delta f(y_1, y_2)| < h, |d(t)| < d. (h > 0, d > 0)$$

Definition 1 Consider the error dynamical system (5). If there exists a constant $T = T(e(0)) > 0$ such that $\lim_{t \rightarrow T} \|e(t)\| = \lim_{t \rightarrow T} \|y(t) - x(t)\| = 0$ and $\|e(t)\| = 0, t > T$, then states of the system (5) will converge to zero in the finite time T .

The control goal of this paper is to design a suitable nonsingular terminal sliding mode controller for stabilization of the uncertain system (5) around zero.

II. MAIN RESULTS

To realize the aforementioned procedure, the terminal sliding mode can be defined

$$s(t) = c e_1(t) + |e_2(t)|^{q/p} \text{sgn } e_2(t) \quad (6)$$

Where $c > 0$, both p and q are positive odd integers, $1 < q/p < 2$.

Theorem 1 Consider the error dynamics (5), if the TSM is designed as (6), then the system state trajectory starting from any initial state $e_1(t_r) \neq 0$ will converge to zero in finite time t_s , given by $t_s = \frac{q}{(q-p)c^{p/q}} |e_1(t_r)|^{1-p/q}$.

Where t_r is the time of synchronous error state trajectory reaching to the sliding surface $s(t) = 0$.

Proof Consider the following Lyapunov function: $V = \frac{1}{2} e_1^2$.

When the sliding surface $s(t) = 0$ is reached, from (6), we have

$$e_1(t) = -\frac{1}{c} |e_2(t)|^{q/p} \text{sgn } e_2(t)$$

Then $\dot{V} = e_1 \dot{e}_1 = e_1 e_2 = -\frac{1}{c} |e_2(t)|^{q/p+1} \leq 0$, only when $e_2 = 0$, $\dot{V} = 0$.

From $s(t) = 0$, one can obtain $e_1 = 0$. Furthermore, from (6), we have

$$\dot{e}_1(t) + c^{p/q} (e_1(t))^{p/q} = 0 \quad (7)$$

Where $e_1(t) = 0$ is the terminal attractor of the system (7).

The finite time t_s that is taken to travel from $e_1(t_r) \neq 0$ to $e_1(t_r + t_s) = 0$ is given by $t_s = -\frac{1}{c^{p/q}} \int_{e_1(t_r)}^{e_1(t_r+t_s)} \frac{de_1(t)}{[e_1(t)]^{p/q}}$
 $= -\frac{1}{c^{p/q}} \int_{e_1(t_r)}^0 \frac{de_1(t)}{[e_1(t)]^{p/q}} = \frac{q}{(q-p)c^{p/q}} |e_1(t_r)|^{1-p/q}$.

This completes the proof.

Having established the suitable sliding surface, the next step is to determine an input signal $u(t)$ to guarantee that the error system trajectories reach to the sliding surface $s(t) = 0$ and stay on it forever. Therefore, a sliding mode controller is proposed based on a double power reaching law [15] as:

$$\begin{aligned} u(t) = & c_1 e_2 + \alpha^2 \phi_1(x_1, y_1) + c_2 \phi_2(x_2, y_2) \\ & - (\beta + f \sin \omega t) \phi_3(x_1, y_1) - \frac{cp}{q} |e_2|^{2-q/p} \text{sgn}(e_2) \\ & - (h + d) \text{sgn}(s) - (k_1 |s|^\lambda + k_2 |s|^\mu) \text{sgn}(s) \end{aligned} \quad (8)$$

Where $k_1 > 0, k_2 > 0, \lambda > 1, 0 < \mu < 1$.

Theorem 2 For system (5), if the controller is designed as (8), then system trajectories reach to the sliding surface in finite time.

Proof Consider the following Lyapunov function: $V = \frac{1}{2} s^2$.

Calculating the derivation along system (5) yields, from (6) and (8).

$$\begin{aligned} \dot{V} = s \dot{s} = & s \left(c e_2 + \frac{q}{p} |e_2|^{q/p-1} \dot{e}_2 \right) \\ = & s [c e_2 + \frac{q}{p} |e_2|^{q/p-1} (-c_1 e_2 - \alpha^2 \phi_1 - c_2 \phi_2 + (\beta + f \sin \omega t) \phi_3 \\ & + \Delta f + d(t) + u(t))] \\ = & s [c e_2 + \frac{q}{p} |e_2|^{q/p-1} (-c_1 e_2 - \alpha^2 \phi_1 - c_2 \phi_2 + (\beta + f \sin \omega t) \phi_3 \\ & + \Delta f + d(t) + c_1 e_2 + \alpha^2 \phi_1 + c_2 \phi_2 - (\beta + f \sin \omega t) \phi_3 \\ & - \frac{cp}{q} |e_2|^{2-q/p} \text{sgn}(e_2) - (h + d) \text{sgn}(s) - (k_1 |s|^\lambda + k_2 |s|^\mu) \text{sgn}(s))] \\ = & \frac{q}{p} |e_2|^{(q-p)/p} s \left[\Delta f + d(t) - (h + d) \text{sgn}(s) - (k_1 |s|^\lambda + k_2 |s|^\mu) \text{sgn}(s) \right] \\ \leq & -\frac{q}{p} |e_2|^{(q-p)/p} (k_1 |s|^{\lambda+1} + k_2 |s|^{\mu+1}) \leq 0 \end{aligned}$$

When $e_2 \neq 0$, $-\frac{q}{p} |e_2|^{(q-p)/p} (k_1 |s|^{\lambda+1} + k_2 |s|^{\mu+1}) < 0$, this is $\dot{V} < 0$.

Then for $e_2 = 0$, it is obtained $s(t) = c e_1(t)$. By substituting (8) into (5), we have

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = \Delta f + d(t) - (h + d) \text{sgn}(s) - (k_1 |s|^\lambda + k_2 |s|^\mu) \text{sgn}(s) \end{cases}$$

So, there are $\dot{e}_2 \leq -k_1 |s|^\lambda - k_2 |s|^\mu$ for $s > 0$ and $\dot{e}_2 \geq k_1 |s|^\lambda + k_2 |s|^\mu$ for $s < 0$.

Therefore, it is concluded that the sliding mode can be reached from anywhere in the phase plane in finite time. Hence the proof is completed.

Remark 1 According to theorem 1 and theorem 2, the terminal sliding mode control law (8) and the terminal sliding surface (6) can make the response system reach the drive system in finite time.

IV SIMULATION RESULTS

To verify the effectiveness of the proposed finite-time controller in the synchronization of two chaotic gyros with uncertainty and external disturbance, a numerical simulation is performed. The system uncertainty and external disturbance are assumed to be $\Delta f(y_1, y_2) = -0.1 \sin(y_1)$, $d(t) = 0.2 \cos(\pi t)$. Respectively, The positive constants are chosen as $\alpha^2 = 100$, $\beta = 1$, $\lambda = 2$, $\mu = 0.5$, $c_1 = 0.5, c_2 = 0.05$, $\omega = 2, f = 35.5, c = 6, p = 3, q = 5, k_1 = 10, k_2 = 6, h = 0.1, d = 0.2$.

The time responses of synchronous error states are depicted in Fig. 2. We can see all the error states quickly converge to zero. As shown in Fig. 3, it is apparent that the trajectories of the response system quickly attain those of the drive system.

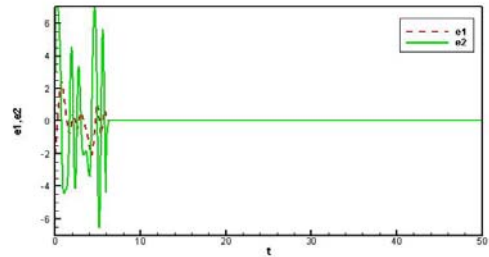
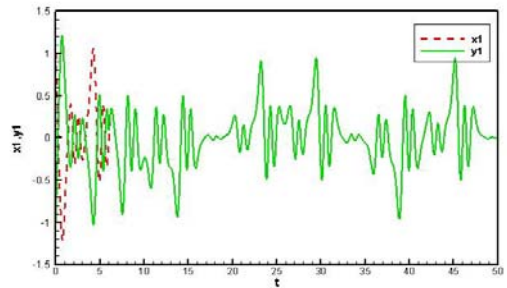


FIGURE II. SYNCHRONIZATION ERROR IN TWO CHAOTIC SYSTEMS.



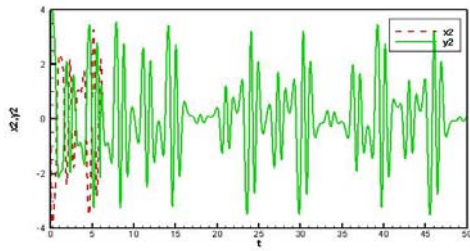


FIGURE III. STATE TRAJECTORIES OF THE DRIVE SYSTEM AND THE RESPONSE SYSTEM.

V CONCLUSIONS

In this paper, the finite-time synchronization problem between two chaotic gyros with uncertainties and external disturbances via terminal sliding mode control are considered. Based on the Lyapunov stability theorem, a sufficient finite-time synchronization criteria is derived. Finally, Numerical simulations are used to verify the proposed control techniques.

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