

# Comparison of Solution of a Class of Second Order Difference Equation

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**Abstract**—Difference equations is a kind of powerful tool to investigate the rule of natural phenomena, such as, physical problems arising in a wide variety of applications. In this paper, we discuss the nodes of the solutions of a class of second order difference equation.

**Keywords**—comparison theorem; difference equation; solution; nodes

## I. INTRODUCTION

Difference equations is a kind of powerful tool to investigate the rule of natural phenomena, such as, physical problems arising in a wide variety of applications. More attentions are paid to difference equations [1-5]. Cheng and Cho [3] discussed the following second order difference equations.

$$\Delta^2 x(k-1) + p_1(k)x(k) = 0,$$

$$\Delta^2 y(k-1) + p_2(k)y(k) = 0,$$

where  $p(k)$  is a real valued function defined on a set of consecutive integers to be specified later. Motivated by the results given in [1, 2, 3, 4, 5], in this paper, we discuss nodes of solutions of the following second order difference equation

$$\Delta^2 x(k-1) + p_1(k)x^2(k) = 0, \quad (1.1)$$

$$\Delta^2 y(k-1) + p_2(k)y^2(k) = 0. \quad (1.2)$$

Suppose that  $x(k)$  and  $y(k)$ ,  $k \in I_1$ , are respectively nontrivial solutions of the second order difference equations (1.1) and (1.2), and  $p_2(k)y(k) \geq p_1(k)x(k)$  for all  $k \in I_3$ . If  $x(k)$  has two consecutive nodes  $a$  and  $b$  in  $[n_0, m]$ , we prove that  $y(k)$  has a node in  $(a, b]$ .

## II. RESULTS

Throughout this paper,  $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{R}$  denotes the set of real numbers. Let  $I_1 = \{n_0, n_0 + 1, \dots, m\}$ ,  $I_2 = \{n_0, n_0 + 1, \dots, m-1\}$ ,  $I_3 = \{n_0 + 1, n_0 + 2, \dots, m-1\}$ ,  $n_0, m \in \mathbb{N}$ .

Let  $x$  be a real function defined on  $I_1$ . By straight line segments we join the points  $(k, x(k))$ ,  $k \in I_1$  to form a broken line. Then this broken line is the graph of a continuous function  $x^*(t)$ , such that  $x^*(t) = x(t)$  for

$k \in I_1$ . The zeros of  $x^*(t)$  are called the nodes of  $x(t)$ . For the convenience of description, we define a function by

$$W(k) = x(k+1)y(k) - y(k+1)x(k), \quad \forall k \in I_2. \quad (2.1)$$

We state the following theorems, which are similar to the results of Cheng and Cho [3].

**Lemma 1.** Let  $(b-1), b \in I_1$ . Suppose that  $x$  and  $y$  are real functions defined on  $I_1$ ,  $x(b-1) > 0$ ,  $y(b-1) > 0$ , and  $x(k)$  has a node  $\beta$  in  $(b-1, \beta)$ ,  $y(k)$  has not node in  $(b-1, \beta]$ . Then  $W(b-1) < 0$ .

**Proof.** Since  $x(b-1) > 0$  and  $x(k)$  has a node  $\beta$  in  $(b-1, \beta)$ ,  $x(b) < 0$ .

If the function  $y(k)$  has not node in  $(b-1, b)$ , then  $y(b) > 0$ . Since  $x(b-1) > 0$ ,  $x(b) < 0$ ,  $y(b-1) > 0$ ,

$$W(b-1) = x(b)y(b-1) - y(b)x(b-1) < 0.$$

If  $y(k)$  has a node  $\delta$  in  $(b-1, b)$ , Since  $y(k)$  has not node in  $(b-1, \beta]$ , then  $\delta > \beta > (b-1)$ .

By the definitions of nodes and  $W$ , we have

$$\begin{aligned} W(b-1) &= x(b)y(b-1) - y(b)x(b-1) \\ &= \left( \frac{x(b)}{x(b-1)} - \frac{y(b)}{y(b-1)} \right) x(b-1)y(b-1) \\ &= \left( \frac{b-\beta}{b-1-\beta} - \frac{b-\delta}{b-1-\delta} \right) x(b-1)y(b-1) \\ &= \frac{(b-\beta)(b-1-\delta) - (b-\delta)(b-1-\beta)}{(b-1-\beta)(b-1-\delta)} x(b-1)y(b-1) \\ &= \frac{(\beta-\delta)x(b-1)y(b-1)}{(b-1-\beta)(b-1-\delta)}. \end{aligned} \quad (2.2)$$

Since  $x(b-1) > 0$ ,  $y(b-1) > 0$ , and  $\delta > \beta > (b-1)$ ,  $W(b-1) < 0$ .

**Theorem 1.** Let  $x(n_0) \geq y(n_0)$  and  $x(n_0+1)y(n_0) - y(n_0+1)x(n_0) \geq 0$ . Suppose that  $x(k)$  and  $y(k)$ ,  $k \in I_1$ , are positive solutions of the second order difference equations (1.1) and (1.2), respectively. If  $p_2(k)y(k) \geq p_1(k)x(k)$  for  $k \in I_1$ , then  $x(k) \geq y(k)$  for all  $k \in I_1$ .

**Proof.** Using (1.1), (1.2) and (2.1), we have

$$\begin{aligned} \Delta W(k) &= \\ &= x(k+2)y(k+1) - y(k+2)x(k+1) - x(k+1)y(k) + y(k+1)x(k) \\ &= -x(k+1)[y(k+2) + y(k)] + y(k+1)[x(k+2) + x(k)] \\ &= -x(k+1)[y(k+2) - 2y(k+1) + y(k)] \\ &\quad + y(k+1)[x(k+2) - 2x(k+1) + x(k)] \\ &= -x(k+1)[\Delta^2 y(k)] + y(k+1)[\Delta^2 x(k)] \\ &= [p_2(k+1)y(k+1) - p_1(k+1)x(k+1)]x(k+1)y(k+1), \forall k \in I_2, (2.3) \end{aligned}$$

Since  $p_2(k)y(k) \geq p_1(k)x(k)$ , and  $x(k)$  and  $y(k)$  are positive for  $k \in I_1$ ,  $\Delta W(k) \geq 0$  for  $k \in I_2$ .

Furthermore, since  $W(n_0) = x(n_0+1)y(n_0) - y(n_0+1)x(n_0) \geq 0$ , we have

$$W(k) = W(n_0) + \sum_{i=n_0}^k \Delta W(i) \geq 0, \forall k \in I_3. \quad (2.4)$$

Using (2.4), we get

$$\begin{aligned} \Delta \left( \frac{x(k)}{y(k)} \right) &= \frac{x(k+1)}{y(k+1)} - \frac{x(k)}{y(k)} \\ &= \frac{x(k+1)y(k) - y(k+1)x(k)}{y(k)y(k+1)} \geq 0, \quad \forall k \in I_2. \quad (2.5) \end{aligned}$$

By (2.5), the function  $x(k)/y(k)$  is a nondecreasing function on  $k \in I_1$ . Since  $x(n_0) \geq y(n_0)$ , we have

$$1 \leq \frac{x(n_0)}{y(n_0)} \leq \frac{x(n_0+1)}{y(n_0+1)} \leq \dots \leq \frac{x(m)}{y(m)}. \quad (2.6)$$

Obviously,  $x(k) \geq y(k)$  for all  $k \in I_1$ , which is the desired relationship. **Theorem 2.** Suppose that  $x(k)$  and  $y(k)$ ,  $k \in I_1$ , are respectively nontrivial solutions of the second order difference equations (1.1) and (1.2), and  $p_2(k)y(k) \geq p_1(k)x(k)$  for all  $k \in I_3$ . If  $x(k)$  has two consecutive nodes  $a$  and  $b$  in  $[n_0, m]$ , then  $y(k)$  has a node in  $(a, b]$ .

**Proof.**

Let

$$n_0 \leq n_j - 1 \leq a < n_j \leq n_K \leq b \leq n_j - 1 \leq m,$$

$n_j, n_K \in I_1$ . Since  $a$  and  $b$  are two consecutive nodes of  $x(k)$ , without loss of generality we can assume that the broken line  $x^*(t) > 0$  for all  $t \in [n_j, n_K] \subset (a, b]$ . Obviously,  $x(n_j - 1) < 0$ ,  $x(n_j) > 0$ ,  $x(n_K) > 0$ ,  $x(n_K + 1) < 0$ .

If we assume that there is not node of  $y(k)$  in  $(a, b]$ , then the sign of  $y^*(t)$  doesn't change in  $(a, b]$ . Without loss of generality we assume that  $y^*(t) > 0$  in  $(a, b]$ , then  $y(n_j) > 0$ ,  $y(n_K) > 0$ . Using the method of proof in lemma 1, we can obtain that

$$W(n_j - 1) = x(n_j)y(n_j - 1) - y(n_j)x(n_j - 1) > 0, \quad (2.7)$$

and

$$W(n_K) = x(n_K+1)y(n_K) - y(n_K+1)x(n_K) > 0. \quad (2.8)$$

On the other hand, by Theorem 1, we have

$$\Delta W(k) = [p_2(k+1) - p_1(k+1)]x(k+1)y(k+1), \forall k \in I_2, \quad (2.9)$$

From (2.7) and (2.9), we have

$$W(n_K) = W(n_j - 1) + \sum_{j=n_j}^{n_K} \Delta W(j) > 0, \quad (2.10)$$

There is a contradiction between (2.8) and (2.10). So  $y(k)$  has a node in  $(a, b]$ .

### III. SUMMARY

Difference equations is a kind of powerful tool to investigate the rule of natural phenomena, such as, physical problems arising in a wide variety of applications. In this paper, we discuss the solutions of a class of second order difference equation.

Let  $x(n_0) \geq y(n_0)$  and  $x(n_0+1)y(n_0) - y(n_0+1)x(n_0) \geq 0$ . Suppose that  $x(k)$  and  $y(k)$ ,  $k \in I_1$ , are positive solutions of the second order difference equations (1.1) and (1.2), respectively. If  $p_2(k)y(k) \geq p_1(k)x(k)$  for  $k \in I_1$ , then  $x(k) \geq y(k)$  for all  $k \in I_1$ .

Suppose that  $x(k)$  and  $y(k)$ ,  $k \in I_1$ , are respectively nontrivial solutions of the second order difference equations (1.1) and (1.2), and  $p_2(k)y(k) \geq p_1(k)x(k)$  for all  $k \in I_3$ . If  $x(k)$  has two consecutive nodes  $a$  and  $b$  in  $[n_0, m]$ , then  $y(k)$  has a node in  $(a, b]$ .

### ACKNOWLEDGEMENTS

This research was supported by Key Projection of Hechi University of China (No. 2014ZD-N003).

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