A Stochastic Lotka-Volterra Competitive System With Feedback Controls

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***Abstract*—In this paper, we consider an autonomous Lotka-Volterra competitive system with stochastic perturbation and feedback controls. Firstly, we show the existence, the uniqueness and the positivity of the solution. Secondly, under a simple assumption, sufficient conditions for stability in the mean and extinction of each population are established.**

Keywords-feedback controls; competitive system; stochastic perturbation; extinction; stability in the mean

I INTRODUCTION

For the last decades, the classical Lotka-Volterra competition system has been studied extensively. Many excellent results are obtained (see [13, 14]).

In [1], the authors argued that in a situation where the equilibrium is not the desirable one (or affordable) and a smaller value is required, we are required to alter the system structurally by introducing a feedback control variable [2]. This can be implemented by means of a biological control or some harvesting procedure so as to make the population stabilize at a lower value. In 1931 V. Volterra explained the balance between two populations of fish in a closed pond using the theory of feedback. Later, a series of mathematical models have been established to describe the dynamics of feedback control systems.

Gopalsamy and Weng [3] introduced a two species autonomous Lotka-Volterra competitive system with feedback controls.

Where $x_i(t)$ denotes the population density, $u_i(t)$ denotes the feedback control variable, and r_i, a_{ij} c_i, e_i, d_i ($i, j = 1, 2$) are positive constants.

$$\begin{cases} dx_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - c_1u_1(t)]dt \\ dx_2(t) = x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t) - c_2u_2(t)]dt \\ du_1(t) = [-e_1u_1(t) + d_1x_1(t)]dt \\ du_2(t) = [-e_2u_2(t) + d_2x_2(t)]dt. \end{cases} \quad (1)$$

They obtained sufficient conditions for the globally asymptotically stable of system (1).

But, in the real world population systems often subject to environmental perturbations. In many cases, these perturbations should not be neglected, and there are many authors have introduced stochastic population models in order to investigate the effect of environmental noises; see (e.g. [4, 6-10, 12]). For example, Meng Liu, Ke Wang [9] discussed a stochastic competition system

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t)]dt + \sigma_1 x_1(t)dB_1(t) \\ \mathrm{d}x_2(t) = x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t)]dt + \sigma_2 x_2(t)dB_2(t). \end{cases} \quad (2)$$

Where r_i, a_{ij}, σ_i ($i, j = 1, 2$) are positive constants.

However, to this day, no scholar has investigated the dynamic behaviors of the stochastic Lotka-Volterra competitive system with feedback controls. In this paper, we consider a stochastic Lotka-Volterra competitive system with feedback controls. Suppose that the environmental noises mainly affect the growth rate r_i , $r_i = r_i + \sigma_i dB_i(t)$, (see [6-10]), where σ_i^2 denotes the intensity of the noise, and $B_i(t)$ is a standard Brownian motion defined on a complete probability space (Ω, F, P) , $i = 1, 2.$ Then we get following stochastic population system

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - c_1u_1(t)]dt + \sigma_1 x_1(t)dB_1(t) \\ \mathrm{d}x_2(t) = x_2(t)[r_2 - a_{21}x(t) - a_{22}y(t) - c_2u_2(t)]dt + \sigma_2 x_2(t)dB_2(t) \\ du_1(t) = [-e_1u_1(t) + d_1x_1(t)]dt \\ du_2(t) = [-e_2u_2(t) + d_2x_2(t)]dt \end{cases} \quad (3)$$

With initial conditions $x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1, 2.$

We can easy to see that system (3) has no positive equilibrium. Hence, there is an interesting and important question is that of whether system (3) still has some stability around some positive point. In this paper, we shall show it.

II MAIN RESULTS

In this section, we first show that the solution of system (3) is global and positive.

Theorem 2.1 For any given initial value $(x_1(0), x_2(0), u_1(0), u_2(0)) \in R_+^4$, system (3) has a unique positive solution $(x_1(t), x_2(t), u_1(t), u_2(t))$ on $t \ge 0$ and the solution will remain in R_+^4 with probability 1, where

$$R_+^4 = \left\{ (x_1, x_2, u_1, u_2) \in R^4 \mid x_i > 0, u_i > 0, i = 1, 2 \right\}$$

Proof . Define a function $V(x_1, x_2, u_1, u_2) = \sum_{i=1}^{2}(x_i - 1 - \ln x_i) + \sum_{i=1}^{2} e_i^{-1} d_i (u_i - 1 - \ln u_i)$, if $(x_1(t), x_2(t), u_1(t), u_2(t)) \in R_+^4$, we obtain that

$$
\begin{aligned}
dV(x_1,x_2,u_1,u_2) &= (x_1-1)\left[r_1 - a_{11}x_1 - a_{12}x_2 - c_1u_1\right]dt + 0.5\sigma_1^2 dt + (x_1-1)\sigma_1 dB_1(t) \\
&\quad + (x_2-1)\left[r_2 - a_{21}x_1 - a_{22}x_2 - c_2u_2\right]dt + 0.5\sigma_2^2 dt + (x_2-1)\sigma_2 dB_2(t) \\
&\quad - c_1u_1 + c_1 + e_1^{-1}c_1d_1x_1 - d_1x_1u_1^{-1} - c_2u_2 + c_2 + e_2^{-1}c_2d_2x_2 - d_2x_2u_2^{-1} \\
&\le \Big[\left(r_1 + a_{11} + a_{21} + e_1^{-1}c_1d_1\right)x_1 - a_{11}x_1^2 + \left(r_2 + a_{22} + a_{12} + e_2^{-1}c_2d_2\right)x_2 - a_{22}x_2^2 \\
&\quad + 0.5\sigma_1^2 + 0.5\sigma_2^2 + c_1 + c_2\Big]dt + (x_1-1)\sigma_1 dB_1(t) + (x_2-1)\sigma_2 dB_2(t) \\
&\le K + (x_1-1)\sigma_1 dB_1(t) + (x_2-1)\sigma_2 dB_2(t).
\end{aligned}
$$

Where K is a positive constant. By the similar proof of [12, Theorem 2.1], we can obtain the desired assertion.

For simplicity, we introduce the following notation. Let

$$\Delta_1 = (a_{22} + \frac{c_2d_2}{e_2})(r_1 - 0.5\sigma_1^2) - (r_2 - 0.5\sigma_2^2)a_{12}$$

$$\Delta_2 = (a_{11} + \frac{c_1d_1}{e_1})(r_2 - 0.5\sigma_2^2) - (r_1 - 0.5\sigma_1^2)a_{21}$$

$$\Delta = (a_{11} + \frac{c_1d_1}{e_1})(a_{22} + \frac{c_2d_2}{e_2}) - a_{12}a_{21}$$

$$f^* = \limsup_{t\to+\infty} f(t)$$

$$f_* = \liminf_{t\to+\infty} f(t)$$

$$\langle f(t)\rangle = t^{-1}\int_0^t f(s)ds$$

Theorem 2.2 (I) If $r_1 < 0.5\sigma_1^2$ and $r_2 < 0.5\sigma_2^2$, then both x_1 and x_2 go to extinction almost surely (a.s.),ie.

(II) If $r_i > 0.5\sigma_i^2$ and $r_j < 0.5\sigma_j^2$, then x_j goes to extinction a.s. and x_i is stable in time average a.s. $i,j=1,2,\ \ i\neq j$,

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)ds = \frac{(r_i - 0.5\sigma_i^2)e_i}{a_{ii}e_i + c_id_i},$$

$$\lim_{t\to+\infty} t^{-1}\int_0^t u_i(s)ds = \frac{(r_i - 0.5\sigma_i^2)d_i}{a_{ii}e_i + c_id_i} \quad a.s.$$

(III) If $r_1 > 0.5\sigma_1^2$ and $r_2 > 0.5\sigma_2^2$.

(1) Suppose that $\Delta>0$ (It is easy to see that $\Delta_1<0$ and $\Delta_2<0$ cannot simultaneously hold in this case).

(a) If $\Delta_i>0$ and $\Delta_j<0$, then x_j goes to extinction a.s. and x_i is stable in time average a.s., $i,j=1,2,\ \ i\neq j$,

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)ds = \frac{(r_i - 0.5\sigma_i^2)e_i}{a_{ii}e_i + c_id_i}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t u_i(s)ds = \frac{(r_i - 0.5\sigma_i^2)d_i}{a_{ii}e_i + c_id_i} \quad a.s.$$

(b) If $\Delta_1>0$ and $\Delta_2>0$, then both x_1 and x_2 are stable in time average a.s.,

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)ds = \frac{\Delta_i}{\Delta}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t u_i(s)ds = \frac{\Delta_i d_i}{\Delta e_i} \quad a.s.\ i=1,2.$$

(2) Suppose that $\Delta<0$ (It is easy to see that $\Delta_1>0$ and $\Delta_2>0$ cannot simultaneously hold in this case).

(a) If $\Delta_i<0$ and $\Delta_j>0$, then x_i goes to extinction a.s. and x_j is stable in time average a.s., $i,j=1,2,\ \ i\neq j$,

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_j(s)ds = \frac{(r_j - 0.5\sigma_j^2)e_j}{a_{jj}e_j + c_jd_j}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t u_j(s)ds = \frac{(r_j - 0.5\sigma_j^2)d_j}{a_{jj}e_j + c_jd_j} \quad a.s.$$

(b) If $\Delta_1<0$ and $\Delta_2<0$, then x_1 and x_2 not simultaneously go to extinction (a.s).

III PROOF OF THE THEOREM 2.2

Firstly, we introduce some fundamental lemmas which will be used.

Lemma 3.1 ([7]). Suppose that $z(t)\in C(\Omega\times R_+)$,

(i) If there exist two positive constants T, and u such that

$$\ln z(t) \le \lambda t - u\int_0^t z(s)ds + \sum_{i=1}^n \alpha_i B_i(t)$$

For all $t\ge T$, where α_i ($i=1,2$) are constants, then

$$\begin{cases} \langle z(t)\rangle^* \le u^{-1}\lambda,\ a.s.\ if\ \ \lambda\ge 0 \\ \lim_{t\to\infty} z(t) = 0,\ a.s.\ if\ \ \lambda<0. \end{cases} \tag{4}$$

(ii) If there exist three positive constants λ, T and u such that $\ln x(t) \ge \lambda t - u\int_0^t x(s)ds + \sum_{i=1}^n \alpha_i B_i(t)$, for all $t\ge T$, then $\langle z(t)\rangle_* \ge u^{-1}\lambda,\ a.s.$.

Lemma 3.2 (see [11]). Consider one-dimensional stochastic differential equation

$$dx = x[a - bx]dt + \sigma x dB(t) \tag{5}$$

where a, b, σ are positive, and B(t) is standard Brownian motion, Under the condition $a>0.5\sigma^2$, for any initial value $x_0>0$, the solution x(t) to (5) has the properties $\lim_{t\to+\infty} t^{-1}\ln x(t) = 0,\quad a.s.$

Lemma 3.3 If $r_i \neq 0.5\sigma_i^2$, for any initial value $(x_1(0),x_2(0),u_1(0),u_2(0))\in R_+^4$, for $x_i(t),u_i(t)$, of system (3) have $\left[t^{-1}\ln x_i(t)\right]^* \le 0,\ \ \lim_{t\to+\infty} t^{-1}u_i(t) = 0,\ \ a.s.\ i=1,2.$

Proof. From (3), it is obvious that $dx_i(t) \le x_i(t)[r_i - a_{ii}x_i(t)]dt + \sigma_i x_i(t)dB_i(t)$.

Denote $X_i(t)$ is the solution to the following stochastic equation

$$\begin{cases} dX_i(t) = X_i(t)[r_i - a_{ii}X_i(t)]dt + \sigma_i X_i(t)dB_i(t) \\ X_i(0) = x_i(0) \quad i=1,2; \end{cases}$$

Then $X_i(t)$ have the following explicit representations respectively

$$X_i(t) = \frac{\exp\{r_it - 0.5\sigma_i^2t + \sigma_iB_i(t)\}}{x_i^{-1}(0) + a_{ii}\int_0^t \exp\{r_is - 0.5\sigma_i^2s + \sigma_iB_i(s)\}ds}, \quad i=1,2. \tag{6}$$

From the third and fourth equations of system (3), we can obtain $u_i(t)$ have the following explicit representations respectively $u_i(t)=\exp\{-e_i t\}[d_i\int_0^t x(s)\exp\{e_i s\}ds+u_i(0)]$.

If $r_i<0.5\sigma_i^2$, then from (6) we have

$$x_i(t)\le X_i(t)\le X_i(0)\exp\{-[0.5\sigma_i^2-r_i-\sigma_i t^{-1}B_i(t)]t\}$$

Because of $\lim_{t\to+\infty} t^{-1}B_i(t)=0\ a.s.$, then $\lim_{t\to+\infty} x_i(t)=0$ and $\left[t^{-1}\ln x_i(t)\right]^*\le 0\ a.s.$, obviously, $\lim_{t\to+\infty} t^{-1}u_i(t)=0\ \ a.s.\ \ i=1,2$. If $r_i>0.5\sigma_i^2$, by virtue of Lemma 3.2 we have

$$\lim_{t\to+\infty} t^{-1}\ln x_i(t)\le \lim_{t\to+\infty} t^{-1}\ln X_i(t)=0\ a.s.\ i=1,2.$$

For $\exp(e_i t)\ln X_i(t)$ apply Itô's formula, we can see that

$$a_{ii}\int_0^t \exp(e_i s)X_i(s)ds=\ln X_i(0)-\exp(e_i t)\ln X_i(t)+e_i\int_0^t\exp(e_i s)\ln X_i(s)ds$$
$$+\int_0^t\exp(e_i s)(r_i-0.5\sigma_i^2)ds+\sigma_i\int_0^t\exp\{e_i s\}dB_i(s).$$

Using mean value theorem of integrals, we obtain

$$\frac{a_{ii}\int_0^t\exp(e_i s)X_i(s)ds}{t\exp(e_i t)}=\frac{\ln X_i(0)}{t\exp(e_i t)}-\frac{\ln X_i(t)}{t}+\frac{e_i\ln X_i(\tau_1)\int_0^t\exp(e_i s)ds}{t\exp(e_i t)}$$
$$+\frac{\int_0^t\exp(e_i s)(r_i-0.5\sigma_i^2)ds}{t\exp(e_i t)}+\frac{\sigma_i[\exp\{e_i t\}B_i(t)-e_iB_i(\tau_2)\int_0^t\exp\{e_i s\}ds]}{t\exp(e_i t)}$$
$$\le t^{-1}\ln X_i(0)-t^{-1}\ln X_i(t)+t^{-1}\ln X_i(\tau_1)-t^{-1}\exp\{-e_i t\}\ln X_i(\tau_1)$$
$$+(e_i t)^{-1}(r_i-0.5\sigma_i^2)+t^{-1}\sigma_iB_i(t)-t^{-1}\sigma_iB_i(\tau_2)+t^{-1}\exp\{-e_i t\}\sigma_iB_i(\tau_2)$$

Where $\tau_i\in[0,t],\ \ i=1,2$. By virtue of Lemma 3.2 and $\lim_{t\to+\infty} t^{-1}B_i(t)=0$, then we can get

$$0\le\lim_{t\to+\infty}\frac{d_i\int_0^t\exp(e_i s)x_i(s)ds+u_i(0)}{t\exp(e_i t)}\le\lim_{t\to+\infty}\frac{d_i\int_0^t\exp(e_i s)X_i(s)ds+u_i(0)}{t\exp(e_i t)}=0\ \ a.s.$$

Therefore, $\lim_{t\to+\infty} t^{-1}u_i(t)=0\ \ a.s.\ \ i=1,2.$

Proof of Theorem 2.2. Applying Itô's formula to (3), we can see that

$$t^{-1}\ln x_1(t)/x_1(0)=(r_1-0.5\sigma_1^2)-a_{11}\langle x_1(t)\rangle-a_{12}\langle x_2(t)\rangle-c_1\langle u_1(t)\rangle+t^{-1}\sigma_1B_1(t)\quad(7)$$
$$t^{-1}\ln x_2(t)/x_2(0)=(r_2-0.5\sigma_2^2)-a_{21}\langle x_1(t)\rangle-a_{22}\langle x_2(t)\rangle-c_2\langle u_2(t)\rangle+t^{-1}\sigma_2B_2(t)\quad(8)$$
$$t^{-1}[u_1(t)-u_1(0)]=-e_1\langle u_1(t)\rangle+d_1\langle x_1(t)\rangle\quad(9)$$
$$t^{-1}[u_2(t)-u_2(0)]=-e_2\langle u_2(t)\rangle+d_2\langle x_2(t)\rangle\quad(10)$$

(I): Suppose that $r_1<0.5\sigma_1^2$ and $r_2<0.5\sigma_2^2$, by virtue of Lemma 3.3's proof we can obtain $\lim_{t\to+\infty} x_i(t)=0, \lim_{t\to+\infty} u_i(t)=0\ \ a.s.\ \ i=1,2$.

(II): Without loss of generality, Suppose that $r_1>0.5\sigma_1^2$ and $r_2<0.5\sigma_2^2$. Since $r_2<0.5\sigma_2^2$, then by (I) we can get, $\lim_{t\to+\infty} x_2(t)=0$ and $\lim_{t\to+\infty} u_2(t)=0.$ for arbitrary $\varepsilon>0$, There exist a positive constant T such that for t > T,

$$-\varepsilon/3\le a_{12}\langle x_2(t)\rangle\le\varepsilon/3$$
$$-\varepsilon/3\le t^{-1}\ln x_2(0)\le\varepsilon/3$$
$$-\varepsilon/3\le c_1(e_1t)^{-1}[u_1(t)-u_1(0)]\le\varepsilon/3$$

Thus computing $(7)-e_1^{-1}c_1d_1\times(9)$, we have

$$t^{-1}\ln x_1(t)\le(r_1-0.5\sigma_1^2+\varepsilon)-(a_{11}+e_1^{-1}c_1d_1)\langle x_1(t)\rangle+t^{-1}\sigma_1B_1(t)\quad(11)$$
$$t^{-1}\ln x_1(t)\ge(r_1-0.5\sigma_1^2-\varepsilon)-(a_{11}+e_1^{-1}c_1d_1)\langle x_1(t)\rangle+t^{-1}\sigma_1B_1(t)\quad(12)$$

Since $r_1>0.5\sigma_1^2$, we can choose ε sufficiently small such that $r_1-0.5\sigma_1^2-\varepsilon>0$,

Apply (i) and (ii) of Lemma 3.1 to above inequalities, we obtain

$$\frac{(r_1-0.5\sigma_1^2-\varepsilon)e_1}{a_{11}e_1+c_1d_1}\le\langle x_1(t)\rangle_*\le\langle x_1(t)\rangle^*\le\frac{(r_1-0.5\sigma_1^2+\varepsilon)e_1}{a_{11}e_1+c_1d_1}\quad a.s.$$

An application of the arbitrariness of ε leads to

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)ds=\frac{(r_1-0.5\sigma_1^2)e_1}{a_{11}e_1+c_1d_1},\ \lim_{t\to+\infty} t^{-1}\int_0^t u_1(s)ds=\frac{(r_1-0.5\sigma_1^2)d_1}{a_{11}e_1+c_1d_1}\quad a.s.$$

Now let us prove (III), computing $(7)-\frac{c_1}{e_1}\times(9)$ and $(8)-\frac{c_2}{e_2}\times(10)$ we have

$$t^{-1}\ln x_1(t)/x_1(0)-(e_1t)^{-1}c_1[u_1(t)-u_1(0)]=(r_1-\frac{\sigma_1^2}{2})-(a_{11}+\frac{c_1d_1}{e_1})\langle x_1\rangle-a_{12}\langle x_2\rangle+t^{-1}\sigma_1B_1(t)\quad(13)$$
$$t^{-1}\ln x_2(t)/x_2(0)-(e_2t)^{-1}c_2[u_2(t)-u_2(0)]=(r_2-\frac{\sigma_2^2}{2})-(a_{22}+\frac{c_2d_2}{e_2})\langle x_2\rangle-a_{21}\langle x_1\rangle+t^{-1}\sigma_2B_2(t)\quad(14)$$

for arbitrary $\varepsilon>0$, There exist a positive constant T, such that for t > T,

$$-\varepsilon/2\le t^{-1}\ln x_i(0)\le\varepsilon/2,\ \ -\varepsilon/2\le c_i(e_it)^{-1}[u_i(t)-u_i(0)]\le\varepsilon/2,\ \ i=1,2\quad(15)$$

Substituting the above inequalities into (13) and (14), one can obtain that for $t>T$, we have

$$t^{-1}\ln x_1(t)\le(r_1-0.5\sigma_1^2+\varepsilon)-(a_{11}+e_1^{-1}c_1d_1)\langle x_1(t)\rangle-a_{12}\langle x_2(t)\rangle+t^{-1}\sigma_1B_1(t)\quad(16)$$
$$t^{-1}\ln x_1(t)\ge(r_1-0.5\sigma_1^2-\varepsilon)-(a_{11}+e_1^{-1}c_1d_1)\langle x_1(t)\rangle-a_{12}\langle x_2(t)\rangle+t^{-1}\sigma_1B_1(t)\quad(17)$$
$$t^{-1}\ln x_2(t)\le(r_2-0.5\sigma_2^2+\varepsilon)-(a_{22}+e_2^{-1}c_2d_2)\langle x_2(t)\rangle-a_{21}\langle x_1(t)\rangle+t^{-1}\sigma_2B_2(t)\quad(18)$$
$$t^{-1}\ln x_2(t)\ge(r_2-0.5\sigma_2^2-\varepsilon)-(a_{22}+e_2^{-1}c_2d_2)\langle x_2(t)\rangle-a_{21}\langle x_1(t)\rangle+t^{-1}\sigma_2B_2(t).\quad(19)$$

Computing $(16)\times(a_{22}+e_2^{-1}c_2d_2)-(19)\times a_{12}$, by virtue of Lemma 3.3, we can obtain

$$(a_{22}+e_2^{-1}c_2d_2)t^{-1}\ln x_1(t)\le(a_{22}+e_2^{-1}c_2d_2)(r_1-0.5\sigma_1^2+\varepsilon)-a_{12}(r_2-0.5\sigma_2^2-\varepsilon)\quad(20)$$
$$-\Delta\langle x_1(t)\rangle+(e_2t)^{-1}(a_{22}e_2+c_2d_2)\sigma_1B_1(t)-t^{-1}a_{21}\sigma_2B_2(t),$$

For t > T, at the same time, computing $(18)\times(a_{11}+e^{-1}c_1d_1)-(17)\times a_{21}$, by virtue of Lemma 3.3, we can obtain

$$(a_{11}+e^{-1}c_1d_1)t^{-1}\ln x_2(t)\le(a_{11}+e^{-1}c_1d_1)(r_2-0.5\sigma_2^2+\varepsilon)-a_{21}(r_1-0.5\sigma_1^2-\varepsilon) \quad (21)$$
$$-\Delta\langle x_2(t)\rangle+(e_1t)^{-1}(a_{11}e_1+c_1d_1)\sigma_2B_2(t)-t^{-1}a_{21}\sigma_1B_1(t).$$

Case (1) :(a) Without loss of generality. Suppose $\Delta_1>0$ and $\Delta_2<0$, then let ε be sufficiently small such that $(a_{11}+e_1^{-1}c_1d_1)(r_2-0.5\sigma_2^2+\varepsilon)-(r_1-0.5\sigma_1^2-\varepsilon)a_{21}<0$, applying

(i) of lemma 3.1 to (21) have $\lim_{t\to+\infty}x_2(t)=0, \lim_{t\to+\infty}u_2(t)=0$ a.s. then the proof of

$$\lim_{t\to+\infty}t^{-1}\int_0^t x_1(s)ds=\frac{(r_1-0.5\sigma_1^2)e_1}{a_{11}e_1+c_1d_1},\quad \lim_{t\to+\infty}t^{-1}\int_0^t u_1(s)ds=\frac{(r_1-0.5\sigma_1^2)d_1}{a_{11}e_1+c_1d_1}\quad a.s.$$

is similar to that of (II) and hence is omitted.

(b) Suppose $\Delta_1>0$ and $\Delta_2>0$. Since $\Delta_1>0$, by virtue of Lemma 3.1 (i), (20) and the arbitrariness of ε, we can obtain

$$\langle x_1(t)\rangle^*\le\frac{(a_{22}+e_2^{-1}c_2d_2)(r_1-0.5\sigma_1^2)-a_{12}(r_2-0.5\sigma_2^2)}{\Delta}=\frac{\Delta_1}{\Delta},\quad a.s. (22)$$

Similarly, by virtue of (21), Lemma 3.1 (i) and the arbitrariness of ε, we have

$$\langle x_2(t)\rangle^*\le\frac{\Delta_2}{\Delta}\quad a.s. \quad (23)$$

Let ε sufficiently small such that $(a_{11}+\frac{c_1d_1}{e_1})\frac{\Delta_1}{\Delta}-\varepsilon>0.$ when (15) and (23) are used in (17), we have

$$t^{-1}\ln x_1(t)\ge(r_1-0.5\sigma_1^2-\varepsilon)-a_{12}\frac{\Delta_2}{\Delta}-(a_{11}+\frac{c_1d_1}{e_1})\langle x_1(t)\rangle+t^{-1}\sigma_1B_1(t)$$
$$\ge(a_{11}+\frac{c_1d_1}{e_1})\frac{\Delta_1}{\Delta}-\varepsilon-(a_{11}+\frac{c_1d_1}{e_1})\langle x_1(t)\rangle+t^{-1}\sigma_1B_1(t).$$

Applying (ii) of lemma 3.1 to above inequality and making use of the arbitrariness of ε, we have

$$\langle x_1(t)\rangle_*\ge\frac{\Delta_1}{\Delta}\qquad a.s. \quad (24)$$

Similarly, leads to $\langle x_2(t)\rangle_*\ge\frac{\Delta_2}{\Delta}$ a.s. together with (22)-(24), have

$$\lim_{t\to+\infty}\langle x_i\rangle=\frac{\Delta_i}{\Delta},\quad a.s.\ \lim_{t\to+\infty}\langle u_i\rangle=\frac{\Delta_i d_i}{\Delta e_i},\quad a.s.\ i=1,2;$$

Case (2): (a) without loss of generality. Suppose $\Delta_1<0$ and $\Delta_2>0$.

Computing $(19)\times(a_{11}+\frac{c_1d_1}{e_1})-(16)\times a_{21}$, have

$$(a_{11}+\frac{c_1d_1}{e_1})t^{-1}\ln x_2(t)\ge t^{-1}a_{21}\ln x_1(t)+(a_{11}+\frac{c_1d_1}{e_1})(r_2-0.5\sigma_2^2-\varepsilon)-a_{21}(r_1-0.5\sigma_1^2+\varepsilon) \quad (25)$$
$$-\Delta\langle x_2(t)\rangle+(a_{11}e_1+c_1d_1)\sigma_2(te_1)^{-1}B_2(t)-a_{21}\sigma_1t^{-1}B_1(t).$$

Let ε be sufficiently small such that $(a_{11}+\frac{c_1d_1}{e_1})(r_2-0.5\sigma_2^2-\varepsilon)-a_{21}(r_1-0.5\sigma_1^2+\varepsilon)>0,$ we can obtain from (25)

$$-a_{21}\left[t^{-1}\ln x_1(t)\right]^*\ge(a_{11}+e_1^{-1}c_1d_1)(r_2-0.5\sigma_2^2-\varepsilon)-a_{21}(r_1-0.5\sigma_1^2+\varepsilon)>0$$

Make use of the arbitrariness of ε and Lemma 3.3, we can obtain

$$\lim_{t\to+\infty}x_1(t)=0,\quad \lim_{t\to+\infty}u_1(t)=0,\quad a.s.$$

At the same time, the similar proof of (II), we can obtain

$$\lim_{t\to+\infty}t^{-1}\int_0^t x_2(s)ds=\frac{(r_2-0.5\sigma_2^2)e_2}{a_{22}e_2+c_2d_2},\quad \lim_{t\to+\infty}t^{-1}\int_0^t u_2(s)ds=\frac{(r_2-0.5\sigma_2^2)d_2}{a_{22}e_2+c_2d_2}\quad a.s.$$

So we omitted it in here.

(b) According to (19), we can see

$$a_{21}\langle x_1(t)\rangle+\left(a_{22}+e_2^{-1}c_2d_2\right)\langle x_2(t)\rangle+t^{-1}\ln x_2(t)\ge(r_2-0.5\sigma_2^2-\varepsilon)+t^{-1}\sigma_2B_2(t)$$

by virtue of $\left[t^{-1}\ln x_2(t)\right]^*\le0$, we can obtain

$$a_{21}\langle x_1(t)\rangle^*+\left(a_{22}+e_2^{-1}c_2d_2\right)\langle x_2(t)\rangle^*\ge(r_2-0.5\sigma_2^2)>0 \quad (26)$$

So the fact that is x_1 and x_2 not simultaneously go to extinction from (26).

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