Stability Analysis of an N-Unit Series Repairable System with a Repairman Doing Other Work

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Abstract—We investigate an N-unit series repairable system with a repairman doing other work. By analysing the spectral distribution of the system operator and taking into account the irreducibility of the semigroup generated by the system operator we show that the dynamic solution converges strongly to the steady state solution. Thus we obtain asymptotic stability of the dynamic solution.

Keywords-series repairable system; \( C_0 \)-semigroup; dynamic solution; asymptotic stability

I. INTRODUCTION

In [1], Liu and Tang studied an N-unit series repairable system with a repairman doing other work, and obtained the expression of Laplace transforms of some primary reliability indices of the system by using the supplementary variable method, the generalized Markov progress method and the Laplace-transform technique. In [2], we proved the well-posedness and the existence of a unique positive dynamic solution of the system, by using \( C_0 \)-semigroup theory of linear operators from [3] and [4]. In this paper, we study the asymptotic stability of the dynamic solution. We first reformulate the system as an abstract Cauchy problem as in [2], and then, we prove that the time-dependent solution converging to its static solution in the sense of the norm through studying the spectrum of the operator and irreducibility of the corresponding semigroup, thus we obtain the asymptotic stability of the time-dependent solution of this system.

The system can be described by the following equations (see [1]):

\[
\begin{align*}
\frac{dp_i(t)}{dt} &= (c + \mu_i)p_i(t) + \int_0^t \mu(x)p_i(t, x)dx + \sum_{j=1}^n \mu_j p_{ij}(t, x)dx, \\
\frac{\partial p_{ij}(t, x)}{\partial t} + \frac{\partial p_{ij}(t, x)}{\partial x} &= -\mu(x)p_{ij}(t, x), \quad i, j = 1, 2, \ldots, n, \\
\frac{\partial p_{ij}(t, x)}{\partial t} + \frac{\partial p_{ij}(t, x)}{\partial x} &= -(c + \mu_i)p_{ij}(t, x), \\
\frac{\partial p_{ij}(t, x)}{\partial t} + \frac{\partial p_{ij}(t, x)}{\partial x} &= -\mu(x)p_{ij}(t, x) + \delta_{ij}p_{ii}(t, x), \quad i, j = 1, 2, \ldots, n, \\
\frac{\partial p_{ij}(t, x)}{\partial t} + \frac{\partial p_{ij}(t, x)}{\partial x} &= -(\lambda + \mu(x))p_{ij}(t, x) + \nu_{ij}p_{ij}(t, x),
\end{align*}
\]

Where \( \Lambda = \sum_{i=1}^n \lambda_i \) , it’s the boundary condition

\[
\begin{align*}
p_i(0, t) &= \lambda_i p_i(t) + \int_0^t \mu(x)p_{i}(t, x)dx, \quad i = 1, 2, \ldots, n, \\
\end{align*}
\]

\( (BC) \)
\[
\begin{align*}
p_i(t, 0) &= \mu(x)p_i(t, x)dx, \\
p_i(0, t) &= 0, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

And its initial condition

\[
\begin{align*}
p_i(0) &= 0, \quad i = 1, 2, \ldots, n, \\
p_i(0, 0) &= 0, \quad i = 1, 2, \ldots, n, \\
p_i(0, x) &= 0, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

Where \( (t, x) \in [0, +\infty) \times [0, +\infty) \) , \( (t, y) \in [0, +\infty) \times [0, +\infty) \) , \( p_i(t) \) gives the probability that at time \( t \) all the units are in working state and the repairman is idle; Analogously, \( p_i(t) \) represents the probability that at time \( t \) the repairman is repairing the failed unit \( i \) , and the hours that the failed unit has been repaired lies in \( (y, y + dy) \) , \( p_i(t)dx \) represents the probability that at time \( t \) all the units are in working state and repairman is servicing for customer, \( p_i(t)dx \) represents the probability that at time \( t \) all the units are in working state and repairman is servicing for customer, the other customers is waiting for service.

Throughout the paper we require the following assumption for the functions \( \mu(x) \) and \( \mu_i(x) \).
General assumption. The functions $\mu$ and $\mu_i: R \to R^+$ is measurable and bounded such that

$$\mu := \lim_{x \to +\infty} \mu_i(x) > 0, \quad \mu_i(x) := \lim_{x \to +\infty} \mu_i(x) > 0, i = 1, 2, \ldots, n, \quad \mu := \min\{\mu, \mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_n(\cdot)\}$$

II. THE PROBLEM AS AN ABSTRACT CAUCHY PROBLEM

To apply semigroup theory we transform in this section the system $(R_i, (BC_i), (IC_i))$ into an abstract Cauchy problem

[3, Def. II.6.1] on the Banach space $(X, \|\cdot\|)$, where

$$X = C \times \left(\mathbb{L}_+[0, \infty)\right)^n \times \left(\mathbb{L}_+[0, \infty)\right)^{n+2}$$

And

$$\|p\| = |p_0| + \sum_{i=1}^n \|p_i\|_{L^2([0, \infty))} + \sum_{i=1}^n \|p_i\|_{L^2([0, \infty))} + \sum_{i=1}^n \|p_i\|_{L^2([0, \infty))}$$

The Space $(X, \|\cdot\|)$ will also be called state space.

In a first step we introduce a "maximal operator" $(A_m, D(A_m))$ describing only $(R)$. It is given by

$$A_m = \begin{pmatrix} -(c + \Lambda) & \psi_1 & \ldots & \psi_n & \psi & 0 & 0 & \ldots & 0 & 0 \\
0 & D_{11} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & D_{i} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & \lambda_1 & D_{11} & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & \lambda_2 & D_{i} & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & D_{i} & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & D_{i} & \ldots \\
\end{pmatrix}, \quad (1)$$

Here and in the following $\psi, i = 1, 2, \ldots, n$ and $\psi$ denote the linear functionals

$$\psi_i : \mathbb{L}_+[0, \infty) \to C, \quad f \mapsto \psi_i(f) = \int_0^\infty \mu_i(y) f(y) dy$$

$$\psi_i : \mathbb{L}_+[0, \infty) \to C, \quad f \mapsto \psi_i(f) = \int_0^\infty \mu(y) f(x) dx$$

(3)

Moreover, the operators $D_{ii}, i = 1, 2, \ldots, n, D_{ij}, i = 1, 2, \ldots, n$ and $D_{ij}$ on $W^{1,1}[0, \infty)$ are defined respectively as

$$D_{ii} = \frac{d}{dy} - \mu_i(y), i = 1, 2, \ldots, n$$

$$D_{ij} = \frac{d}{dx} - \mu(x), i = 1, 2, \ldots, n$$

(4)

To model the boundary conditions $(BC)$ we use an abstract approach as in [5]. To this purpose we introduce the boundary space $\partial X = C^{2n+2}$ and then define the following boundary operators $L$ and $\Phi$ by

$$L : D(A_m) \to \partial X \quad \Phi : X \to \partial X$$

(5)

And

$$\phi : X \to \partial X$$

$$\phi : X \to \partial X$$

(6)

Now the system operator $(A_n, D(A_n))$ on $X$ given by

$$Ap = A_m p, \quad D(A_n) = \{p \in D(A_n) | \|p\|_{D(A_n)} \leq \|p\|_{X}\}$$

(7)

describes the system completely. The above equations $(R_i, (BC_i)$ and $(IC_i)$ are equivalent to the abstract Cauchy problem in the Banach space $X$ as follows.

$$\begin{cases} \frac{dp(t)}{dt} = Ap(t), t \in [0, \infty) \\
|p(0) = (0, 0, \ldots, 0) \in X \end{cases}$$

(ACP)

We start from the operator $(A_n, D(A_n))$ defined by

$$D(A_n) := \{p \in D(A_n) | \|p\|_{D(A_n)} \leq \|p\|_{X}\} \quad A_n p := A_n p$$

(8)

Lemma 2.1: For $\gamma \in \rho(A_n)$, we have

$$p \in \ker(\gamma - A_n)$$

(9)

$$p = (p_0, p_1(y), p_2(y), \ldots, p_{n-1}(y), p_n(x), p_{n+1}(x), \ldots, p_{n+n}(x), p_{n+n+1}(x), \ldots, p_{2n+n+1}(x)) \in D(A_n)$$

$$p = (p_0, p_1(y), p_2(y), \ldots, p_{n-1}(y), p_n(x), p_{n+1}(x), \ldots, p_{n+n}(x), p_{n+n+1}(x), \ldots, p_{2n+n+1}(x)) \in D(A_n)$$

(10)

$$p = c_i e^{-\gamma t} \begin{pmatrix} p_0 \\
p_0 \\
p_1(y) \\
p_2(y) \\
\vdots \\
p_{n-1}(y) \\
p_n(x) \\
p_{n+1}(x) \\
\vdots \\
p_{n+n}(x) \\
p_{n+n+1}(x) \\
\vdots \\
p_{2n+n+1}(x) \end{pmatrix}, \quad i = 1, 2, \ldots, n$$

(11)
decomposes as
of the
operator
can be represented by the
is surjective,
Moreover, since
is injective,
Using (6, Lemma 1.2), the domain
of the
maximal operator
decouples as

For each \( \lambda \in \sigma(A) \), the operator \( P_{\lambda} \) can be represented by the

For each \( \lambda \in \sigma(A) \), the operator \( P_{\lambda} \) can be represented by the

(1)

(2)

(3)

(4)

(5)

(6)

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(40)
The Following result, which can be found in [7], plays important role for us to obtain our main result in Section 3.

Lemma 2.3(The characteristic equation): Let \( \gamma \in \rho(A) \), then

\[
(i) \quad \gamma \in \sigma(A) \iff 1 \in \sigma(A) \cap \rho(A),
\]

(ii) If \( \gamma \in \rho(A) \) and there exists \( \gamma_0 \in C \) such that \( 1 \in \sigma(A) \cap \rho(A) \), then

\[
\gamma \in \sigma(A) \iff 1 \in \sigma(A) \cap \rho(A).
\]  

III. STABILITY OF THE SOLUTION

In this section, we will investigate the asymptotic stability of the dynamic olution of the system. We show first the following lemmas:

Lemma 3.1: For the operator \( (A, D(A)) \) we have \( 0 \in \sigma(A) \).

Applying Lemma 2.3(ii) we can show that \( 0 \) is the only spectral value of \( A \) on the imaginary axis.

Lemma 3.2: Under the General Assumption 1.1, the spectrum \( \sigma(A) \) of \( A \) satisfies \( \sigma(A) \cap i\mathbb{R} = \{0\} \).

Lemma 3.3: If \( \gamma \in \rho(A) \) and there exists \( \gamma_0 \in C \) such that \( 1 \in \sigma(A) \cap \rho(A) \), then

\[
R(\gamma, A) = R(\gamma, A_0) + D_p (I - D_p)^{-1} \Phi R(\gamma, A_0).
\]  

Lemma 3.4: The semigroup \( (T(t))_{t \geq 0} \) generated by \( (A, D(A)) \) is irreducible.

With this at hand one can then show the convergence of the semigroup to a one dimensional equilibrium point, see [7, Thm. 3.11].

Theorem 3.5: The space \( X \) can be decomposed into the direct sum

\[
X = X_1 \oplus X_2,
\]

where \( X_1 = \{f(x)T(t)\}_{t \geq 0} = \ker A \) is one-dimensional and spanned by a strictly positive eigenvector \( \hat{p} \in \ker A \) of \( A \). In addition, the restriction \( (T(t)|_{X_2})_{t \geq 0} \) is strongly stable.

Corollary 3.7: The dynamic solution of the system (R), (BC) and (IC) converges strongly to the steady-state solution as time tends to infinity, that is,

\[
\lim_{t \to \infty} p(t, .) = \alpha \hat{p},
\]

where \( \alpha > 0 \), \( \hat{p} > 0 \) and \( \hat{p} \) as in Corollary 3.6.

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