Specific Solutions of a Class of Second Order
Difference Equation with Boundary Conditions

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Abstract—Difference equation is a kind of important tool to study
the rule of natural phenomena. In this paper, we discuss several
specific solutions of a class of second order difference equation
with boundary conditions.

Keywords—difference equation; second order; boundary
condition; specific solution

I. INTRODUCTION

Difference equations is a kind of important tool to study the
rule of natural phenomena, such as, physical problems arising
in a wide variety of applications. Cheng and Cho [3]
investigated the following second order difference equations

$$\Delta^2 x(k - 1) + p(k) x(k) = 0,$$

where p(k) is a real valued function defined on a set of the
natural numbers.

Motivated by the results given in [1, 2, 3, 4, 5], in this paper,
we discuss specific solutions of the following second order
difference equations for $k \in \{1, 2, \cdots, N\}$.

$$\Delta^2 x(k - 1) + p(k) x(k) = 0, \quad (1)$$

satisfying

$$x(0) = 0, x(N + 1) = 0,$$

or

$$x(0) + \alpha x(1) = 0, x(N + 1) + \lambda x(N) = 0,$$

or

$$x(0) + \alpha(1) = 0, x(2m) + \lambda x(2m) = 0, x(4m + 1) + \theta x(4m) = 0.$$

II. MAIN RESULTS

Throughout this paper, let n, m be natural numbers,

$$I_{n,m} = \{n, n + 1, \cdots, m\}.$$

Proposition 1. Let $N = 2m + 1$.

$$p(k) = \begin{cases} 0, & k \in I_{0,m}, \\ 2/(m + 1), & k = m + 1, \\ 0, & k \in I_{m+2,N}. \end{cases} \quad (2)$$

and

$$x(k) = \begin{cases} k, & k \in I_{0,m+1}, \\ 2m - k + 2, & k \in I_{m+1,N+1}. \end{cases} \quad (3)$$

Then (3) is the specific solution of second order difference
equation (1).

Proof. From (2) and (3), we have

$$\frac{\Delta^2 x(m)}{x(m + 1)} = \frac{x(m + 2) - 2x(m + 1) + x(m)}{x(m + 1)} = \frac{m - 2(m + 1) + m}{m + 1} = \frac{-2}{m + 1} = -p(m + 1), \quad (4)$$

Since

$$\Delta^2 x(k - 1)/x(k) = 0 = -p(k)$$

for $k \in I_{1, m} \cup k \in I_{m+2,N}$, (3) is a specific solution of (1) with

$$x(0) = 0, x(N + 1) = 0.$$

Proposition 2. Let $N = 2m$.

$$p(k) = \begin{cases} 0, & k \in I_{0,m}, \\ (2m+1)/m(m+1), & k = m + 1, \\ 0, & k \in I_{m+2,N}. \end{cases} \quad (5)$$

and

$$x(k) = \begin{cases} k, & k \in I_{0,m+1}, \\ (m+1)(2m - k + 1)/m, & k \in I_{m+1,N+1}. \end{cases} \quad (6)$$

Then (6) is the specific solution of (1).

Proof. From (5) and (6), we obtain that

$$\Delta^2 x(m)/x(m + 1) = \frac{x(m + 2) - 2x(m + 1) + x(m)}{x(m + 1)} = \frac{(m^2 - 1)/m - 2(m + 1) + m}{m + 1} = \frac{m^2 - 1 - 2m(m + 1) + m^3}{m(m + 1)} = \frac{2m + 1}{m + 1} = -p(m + 1). \quad (7)$$
Since $\Delta^2 x(k-1)/x(k) = 0 = -p(k)$ for $k \in I_{\omega,0} \cup k \in I_{\sigma + 2, N}$, (6) is a specific solution of (1) such that $x(0) = 0$, $x(N+1) = 0$.

**Proposition 3.** Let $N = 2m + 1$.

$$p(k) = \begin{cases} 0, & k \in I_{\omega,0}, \\ \frac{N + 1 + N\lambda + N\sigma + 2m\lambda\sigma}{(m + 1 + m\lambda)(m + 1 + m\lambda)}, & k = m + 1, \\ 0, & k \in I_{\omega + 2, N}, \end{cases}$$

and

$$x(k) = \begin{cases} \frac{1 + (1 + \lambda)(k - 1)}{(m + 1 + m\lambda)(2m + 1 + 1/(1 + \lambda) - k)}, & k \in I_{\omega,0}, \\ \frac{1}{N + 1 + N\lambda + N\sigma + 2m\lambda\sigma}, & k = m + 1, \\ 0, & k \in I_{\omega + 2, N}. \end{cases}$$

Then (9) is the specific solution of (1) with $x(0) + \sigma x(1) = 0$, $x(N+1) + \lambda x(N) = 0$.

**Proof.** From (8) and (9), we conclude

$$\Delta^2 x(m)/x(m+1) = x(m+2) - 2x(m+1)/x(m+1) = (m + 1 + m\lambda)$$

$$= \frac{1 + \sigma}{m + 1 + m\lambda} + \frac{x(m + 1)}{x(m)}$$

$$= \frac{m + 1 + m\lambda}{m + 1 + m\lambda} + \frac{(1 + \lambda)(m + 1 + m\lambda)}{(m + 1 + m\lambda)(m + 1 + m\lambda)}$$

$$= \frac{2m + 2(2m + 1)\lambda + (2m + 1)\sigma + 2m\lambda\sigma}{(m + 1 + m\lambda)(m + 1 + m\lambda)}$$

and

$$\Delta^2 x(k-1)/x(k) = 0 = -p(k)$$

for $k \in I_{\omega,0} \cup k \in I_{\sigma + 2, N}$.

**Proposition 4.** Let $N = 2m$.

$$p(k) = \begin{cases} 0, & k \in I_{\omega,0}, \\ \frac{N + 1 + N\lambda + N\sigma + (N - 1)\lambda\sigma}{(m + (m - 1)\sigma)(m + 1 + m\lambda)}, & k = m, \\ 0, & k \in I_{\omega + 2, N}. \end{cases}$$

and

$$x(k) = \begin{cases} \frac{1 + (1 + \lambda)(k - 1)}{(m + 1 + m\lambda)(2m + 1 + 1/(1 + \lambda) - k)}, & k \in I_{\omega,0}, \\ \frac{1}{N + 1 + N\lambda + N\sigma + (N - 1)\lambda\sigma}, & k = m, \\ 0, & k \in I_{\omega + 2, N}. \end{cases}$$

Then (12) is the solution of (1) with $x(0) + \sigma x(1) = 0$, $x(N+1) + \lambda x(N) = 0$.

**Proof.** Using (11) and (12), we have

$$\Delta^2 x(m)/x(m) = x(m+1) - 2x(m) + (m - 1)/x(m)$$

$$= \frac{x(m+1) - x(m) - (x(m) - x(m+1))}{x(m)}$$

$$= \frac{m(1 + \sigma) - \sigma}{(m + 1)(m + 1)\sigma} + 1 + \sigma$$

$$= \frac{1 + \sigma}{m + (m - 1)\sigma} + \frac{1 + \lambda}{m + 1 + m\lambda}$$

$$= \frac{(1 + \sigma)(m + 1 + m\lambda) + (1 + \lambda)(m + (m - 1)\sigma)}{(m + (m - 1)\sigma)(m + 1 + m\lambda)}$$

$$= \frac{2m + 2m\lambda + 2m\sigma}{(m + (m - 1)\sigma)(m + 1 + m\lambda)}$$

$$= \frac{p(m)}{1 + \sigma}.$$ 

$$\Delta^2 x(k-1)/x(k) = 0 = -p(k)$$

for $k \in I_{\omega,0} \cup k \in I_{\sigma + 2, N}$.

**Proposition 5.** Let

$$p(k) = \begin{cases} 0, & k \in I_{\omega,0}, \\ \frac{2m + 1 + 2m\sigma + (2m - 1)\lambda\sigma}{(m + (m - 1)\sigma)(m + 1 + m\lambda)}, & k = m, \\ 0, & k \in I_{\omega + 2, N}. \end{cases}$$

and

$$x(k) = \begin{cases} \frac{1 + (1 + \lambda)(k - 1)}{(m + (m - 1)\sigma)(m + 1 + m\lambda) - (1 + \lambda)(k - 1)}, & k \in I_{\omega,0}, \\ \frac{1}{(m + (m - 1)\sigma)(m + 1 + m\lambda)}, & k = m, \\ 0, & k \in I_{\omega + 2, N}. \end{cases}$$

Then (15) is the solution of (1) with $x(0) + \sigma x(1) = 0$, $x(N+1) + \lambda x(N) = 0$, $x(4m+1) + \theta x(4m) = 0$.

**Proof.** Using (14) and (15), we have

$$\Delta^2 x(m)/x(m) = \frac{2m + 2m\lambda + 2m\sigma}{(m + (m - 1)\sigma)(m + 1 + m\lambda)} = p(m),$$

$$\Delta^2 x(3m)/x(3m) = \frac{x(3m+1) - 2x(3m) + x(3m-1)}{x(3m)}$$

$$= \frac{x(3m+1) - 2x(3m) + x(3m-1)}{x(3m)}.$$
\[
\begin{align*}
\Delta^2 x(k-1) / x(k) &= 0 = -p(k)
\end{align*}
\]
for \( k \in I_1 \cup I_{m+1} \cup I_{3m+1} \cup I_{4m} \), (15) is a specific solution of (1) with
\[
x(0) + \sigma x(1) = 0, x(2m+1) + \lambda x(2m) = 0, x(4m+1) + \theta x(4m) = 0
\]

III. SUMMARY

Difference equation is a kind of important tool to study the rule of natural phenomena. In this paper, we discuss several specific solutions of a class of second order difference equation with boundary conditions.

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