The Inclusion-Exclusion Principle On the Set of IF-sets

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Abstract

P. Grzegorzewski [3] has worked the probability version of the inclusion-exclusion principle and made a generalization for IF-events. He had applied two versions of the generalized formula, corresponding to different t-conorms and so defined the union of IF-events. This paper contains the generalization of the Grzegorzewski theorem. We prove it for mappings from the set of IF sets to the unit interval([2], [1]). Similar generalizations are presented in [4] and [5].

Keywords: inclusion-exclusion principle, IF-sets, probability

1. Introduction

K. Atanassov introduced in [1] the notion of an IF - set as a mapping

$$A = (\mu_A, \nu_A).$$

$$\mu_A, \nu_A : \Omega \rightarrow \langle 0, 1 \rangle$$

are such that $\mu_a + \nu_a \leq 1$, $\mathcal{F}$ is the set of all IF -sets such that $A = (\mu_A, \nu_A)$. He considered also the following operations on $\mathcal{F}$:

$$\begin{aligned} A \cap B &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) \\ &= (\min(\mu_A, \mu_B), \max(\nu_A, \nu_B)) \end{aligned}$$

$$\begin{aligned} A \cup B &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B) \\ &= (\max(\mu_A, \mu_B), \min(\nu_A, \nu_B)) \end{aligned}$$

for any $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \mathcal{F}$.

P. Grzegorzewski in [3] considers a classical probability space $(\Omega, \mathcal{S}, \mathcal{P})$, when Ω is a non-empty set, $\mathcal{S}$ is a σ -algebra of subsets of Ω , and $P : \mathcal{S} \rightarrow \langle 0, 1 \rangle$ is a probability measure, i.e. P is σ -additive and $P(\Omega) = 1$. He works with IF-events, that are such IF-sets $A = (\mu_A, \nu_A)$ that $\mu_A, \nu_A : \Omega \rightarrow \langle 0, 1 \rangle$ are $\mathcal{S}$ -measureable, i.e. $\mathcal{B} \subset \mathcal{R}, \mathcal{B}$ is a Borel set $\Rightarrow \mu_A^{-1}(\mathcal{B}) \in \mathcal{S}, \nu_A^{-1}(\mathcal{B}) \in \mathcal{S}$.

P. Grzegorzewski considered the mapping $m : \mathcal{F} \rightarrow \langle 0, 1 \rangle$, defined by the equality:

$$\mathcal{P}(A) = \mathcal{P}(\mu_A, \nu_A) = \left(\int_{\Omega} \mu_A dP, 1 - \int_{\Omega} \nu_A dP \right).$$

He extended the inclusion-exclusion principle for such mappings, i.e.

$$\mathcal{P} \left(\bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n \sum_{j_1, \dots, j_k} (-1)^{k+1} m(A_{j_1} \cap \dots \cap A_{j_k}),$$

e.g.

$$\mathcal{P}(A_1 \cup A_2) = \mathcal{P}(A_1) + \mathcal{P}(A_2) - \mathcal{P}(A_1 \cap A_2) \quad (\mathbf{I})$$

or

$$\begin{aligned} \mathcal{P}(A_1 \cup A_2 \cup A_3) &= \mathcal{P}(A_1) + \mathcal{P}(A_2) + \mathcal{P}(A_3) - \\ &- \mathcal{P}(A_1 \cap A_2) - \mathcal{P}(A_1 \cap A_3) - \mathcal{P}(A_2 \cap A_3) + \\ &+ \mathcal{P}(A_1 \cap A_2 \cap A_3) \end{aligned}$$

etc.

In the paper, we prove the inclusion-exclusion principle for any strongly additive mappings $m : \mathcal{F} \rightarrow \langle 0, 1 \rangle$ i.e. mappings satisfying (I). The result is a generalization of the result of [3]. E.g. the inclusion principle works for the mappings $m^b(A), m^\sharp(A) : \mathcal{F} \rightarrow \langle 0, 1 \rangle$ defined by

$$m^b(A) = \frac{1}{2} \int_{\Omega} \mu_A dP + \frac{1}{2} \int_{\Omega} \nu_A dP,$$

$$m^\sharp(A) = \frac{3}{4} \int_{\Omega} \mu_A dP + \frac{1}{4} \int_{\Omega} \nu_A dP,$$

hence also for $\mathcal{P} : \mathcal{F} \rightarrow \langle 0, 1 \rangle \times \langle 0, 1 \rangle$ defined by the equality

$$\mathcal{P}(A) = \left(m^b(A), m^\sharp(A) \right).$$

Of course, the mapping cannot be covered by the Grzegorzewski result. Recall that another generalizations of [1] will be published in [4] and [5].

2. Inclusion-exclusion principle for IF-sets

Theorem 1 Let $\mathcal{F}$ be the set of pairs $A = (\mu_A, \nu_A)$;

$$A \leq B$$

$$\mu_A \leq \mu_B, \nu_A \geq \nu_B$$

$$0 = (0, 1)$$

$\mu_A, \nu_A : \Omega \rightarrow \langle 0, 1 \rangle, \mu_A + \nu_A \leq 1$. Let the mapping $m : \mathcal{F} \rightarrow \langle 0, 1 \rangle$ be strongly additive, that is

$$m(a \cup b) + m(a \cap b) = m(a) + m(b) \quad (1)$$

and

$$m(0) = 0. \quad (2)$$

Then for n even we have

$$m(a_1 \cup a_2 \cup \dots \cup a_n) + \sum_{k=1}^{n/2} S_{2k}^{(n)} = \sum_{k=1}^{n/2} S_{2k-1}^{(n)}, \quad (3)$$

where

$$S_k^{(n)} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} m(a_{i_1} \cap a_{i_2} \cap \dots \cap a_{i_k}).$$

and for n odd we have

$$\begin{aligned} m(a_1 \cup a_2 \cup \dots \cup a_n) + \sum_{k=1}^{(n+1)/2-1} S_{2k}^{(n)} &= \\ &= \sum_{k=1}^{(n+1)/2} S_{2k-1}^{(n)}. \end{aligned} \quad (4)$$

Proof

For the inclusion-exclusion principle on the IF-sets the distributivity law holds:

$$\begin{aligned} (a \cap c) \cup (b \cap c) &= \\ &= ((\mu_a \wedge \mu_c) \vee (\mu_b \wedge \mu_c), (\nu_a \vee \nu_c) \wedge (\nu_b \vee \nu_c)) = \\ &= (\mu_a \vee \mu_b) \wedge \mu_c, (\nu_a \wedge \nu_b) \vee \nu_c = (a \cup b) \cap c. \end{aligned}$$

It holds also:

$$c \cap c = (\mu_c \wedge \mu_c, \nu_c \vee \nu_c) = (\mu_c, \nu_c) = c.$$

2.1. If n is even

For n even the induction assumption is

$$\begin{aligned} &m\left(\bigcup_{k=1}^n a_k\right) + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m(a_{i_1} \cap \dots \cap a_{i_{2k}}) = \\ &= \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n} m(a_{i_1} \cap \dots \cap a_{i_{2k-1}}) \end{aligned} \quad (5)$$

From (1) we have

$$\begin{aligned} m\left(\left(\bigcup_{k=1}^n a_k\right) \cup a_{n+1}\right) + m\left(\left(\bigcup_{k=1}^n a_k\right) \cap a_{n+1}\right) &= \\ &= m\left(\bigcup_{k=1}^n a_k\right) + m(a_{n+1}) \end{aligned} \quad (6)$$

Moreover,

$$m\left(\left(\bigcup_{k=1}^n a_k\right) \cap a_{n+1}\right) = m\left(\bigcup_{k=1}^n (a_k \cap a_{n+1})\right)$$

so we get

$$\begin{aligned} &m\left(\bigcup_{k=1}^n (a_k \cap a_{n+1})\right) + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) = \\ &= \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n} m\left(\bigcap_{j=1}^{2k-1} a_{i_j} \cap a_{n+1}\right). \end{aligned} \quad (7)$$

From (6) and by adding terms to both sides of equation we obtain :

$$\begin{aligned} &m\left(\bigcup_{k=1}^{n+1} a_k\right) + m\left(\bigcup_{k=1}^n (a_k \cap a_{n+1})\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) = \\ &= m(a_{n+1}) + m\left(\bigcup_{k=1}^n a_k\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) \end{aligned} \quad (8)$$

and so for the right side of equation by the induction assumption we have:

$$\begin{aligned} &m(a_{n+1}) + m\left(\bigcup_{k=1}^n a_k\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) \\ &= m(a_{n+1}) + \sum_{k=1}^{n/2} S_{2k-1}^{(n)} + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) \end{aligned} \quad (9)$$

By (8) and (9) we have

$$\begin{aligned} &m\left(\bigcup_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n} m\left(\bigcap_{j=1}^{2k-1} a_{i_j} \cap a_{n+1}\right) = \\ &= m(a_{n+1}) + \sum_{k=1}^{n/2} S_{2k-1}^{(n)} + \\ &+ \sum_{k=1}^{n/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) \end{aligned} \quad (10)$$

hence

$$\begin{aligned}
& m\left(\bigcup_{k=1}^{n+1} a_k\right) + \\
& + \sum_{k=1}^{(n+2)/2-1} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n+1} m\left(\bigcap_{j=1}^{2k} a_{i_j}\right) = \\
& = \sum_{k=1}^{(n+2)/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n+1} m\left(\bigcap_{j=1}^{2k-1} a_{i_j}\right).
\end{aligned}$$

So,

$$m\left(\bigcup_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{(n+2)/2-1} S_{2k}^{(n+1)} = \sum_{k=1}^{(n+2)/2} S_{2k-1}^{(n+1)}.$$

2.2. If n is odd

Let n be odd, hence the induction assumption gives

$$m\left(\bigcup_{k=1}^n a_k\right) + \sum_{k=1}^{(n+1)/2-1} S_{2k}^{(n)} = \sum_{k=1}^{(n+1)/2} S_{2k-1}^{(n)}.$$

Induction assumption implies

$$\begin{aligned}
& m\left(\bigcup_{k=1}^n (a_k \cap a_{n+1})\right) + \\
& + \sum_{k=1}^{(n+1)/2-1} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) = \\
& = \sum_{k=1}^{(n+1)/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n} m\left(\bigcap_{j=1}^{2k-1} a_{i_j} \cap a_{n+1}\right). \quad (11)
\end{aligned}$$

By the same proceeding as in (8), for n odd we have:

$$\begin{aligned}
& m\left(\bigcup_{k=1}^{n+1} a_k\right) + m\left(\bigcup_{k=1}^n (a_k \cap a_{n+1})\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\
& + \sum_{k=1}^{(n+1)/2-1} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right) = \\
& = m\left(\bigcup_{k=1}^n a_k\right) + m(a_{n+1}) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\
& + \sum_{k=1}^{(n+1)/2-1} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right). \quad (12)
\end{aligned}$$

By induction assumption and (11) and (12)

$$\begin{aligned}
& m\left(\bigcup_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \\
& + \sum_{k=1}^{(n+1)/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n} m\left(\bigcap_{j=1}^{2k-1} a_{i_j} \cap a_{n+1}\right) + \\
& = \sum_{k=1}^{n+1/2} S_{2k-1}^{(n)} + m(a_{n+1}) + \\
& \sum_{k=1}^{(n+1)/2-1} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} m\left(\bigcap_{j=1}^{2k} a_{i_j} \cap a_{n+1}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& m\left(\bigcup_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{(n+1)/2} \sum_{1 \leq i_1 < \dots < i_{2k} \leq n+1} m\left(\bigcap_{j=1}^{2k} a_{i_j}\right) = \\
& = \sum_{k=1}^{(n+1)/2} \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n+1} m\left(\bigcap_{j=1}^{2k-1} a_{i_j} \cap a_{n+1}\right).
\end{aligned}$$

$$m\left(\bigcup_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{(n+1)/2} S_{2k}^{(n)} = \sum_{k=1}^{(n+1)/2} S_{2k-1}^{(n)}$$

3. Examples

Example 1 Let $a, b, c \in \mathcal{F}$. Then,

$$m((a \cup b) \cap c) = m((a \cap c) \cup (b \cap c)),$$

hence for $n = 3$

$$m((a \cup b) \cup c) + m((a \cup b) \cap c) = m(a \cup b) + m(c)$$

$$\begin{aligned}
& m(a \cup b \cup c) + m(a \cap c \cup b \cap c) + m(a \cap b) = \\
& = m(a \cup b) + m(c) + m(a \cap b)
\end{aligned}$$

$$m(a \cup b \cup c) + m(a \cap c \cup b \cap c) + m(a \cap b \cap c) + m(a \cap b) = m(a) + m(b) + m(c) + m(a \cap b \cap c)$$

$$\begin{aligned}
& m(a \cup b \cup c) + m(a \cap c) + m(b \cap c) + m(a \cap b) = \\
& = m(a) + m(b) + m(c) + m(a \cap b \cap c)
\end{aligned}$$

4. Conclusions

The classical inclusion- exclusion principle states that for any probability measure $\mathcal{P} : (\Omega, \mathcal{S}) \rightarrow \langle 0, 1 \rangle$ and any $A_1, \dots, A_n \in \mathcal{S}$ holds

$$\mathcal{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathcal{P}(A_i) - \sum_{i < j} \mathcal{P}(A_i \cap A_j) +$$

$$+ \sum_{i < j < k} \mathcal{P}(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} P \left(\bigcap_{i=1}^n A_i \right).$$

For strongly additive measures $m : \mathcal{S} \rightarrow [0, 1]$ it holds:

$$\begin{aligned} m(a_1 \cup a_2 \cup a_3) &= m(a_1) + m(a_2) + m(a_3) \\ &\quad - m(a_1 \cap a_2) - m(a_1 \cap a_3) - m(a_2 \cap a_3) \\ &\quad + m(a_1 \cap a_2 \cap a_3). \text{see [4]} \end{aligned}$$

and similarly for any $m(a_1 \cup a_2 \cup \dots \cup a_n)$. In this paper we generalize the principle for strongly additive states defined on the set of IF-sets.

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