Abstract

A new method to construct aggregation functions is introduced. These aggregation functions are called biconic aggregation functions with a given diagonal (resp. opposite diagonal) section and their construction method is based on linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square and the points (0, 1) and (1, 0) (resp. (0, 0) and (1, 1)). Special classes of biconic aggregation functions such as biconic semi-copulas, quasi-copulas and copulas are studied in detail.

Keywords: Aggregation function, Quasi-copula, Copula, Diagonal section, Opposite diagonal section, Linear interpolation

1. Introduction

A binary aggregation function $A$ is a $[0, 1]^2 \rightarrow [0, 1]$ function satisfying the following conditions:

(i) $A(0, 0) = 0$ and $A(1, 1) = 1$;
(ii) for any $x, x', y, y' \in [0, 1]$ such that $x \leq x'$ and $y \leq y'$, it holds that $A(x, y) \leq A(x', y')$.

Aggregation functions are of great importance in many fields of application. Their most prominent uses is as logical connectives in fuzzy set theory [2].

To increase modelling flexibility, new methods to construct aggregation functions are being proposed continuously in the literature [6, 18].

Special classes of aggregation functions are of particular interest, such as semi-copulas [15, 16], triangular norms [1, 22], quasi-copulas [17, 24] and copulas [1, 25]. They are all conjunctors, in the sense that they extend the classical Boolean conjunction.

Recall that an aggregation function is a semi-copula if it has 1 as neutral element, i.e. $A(x, 1) = A(1, x) = x$ for any $x \in [0, 1]$. Evidently, any semi-copula $S$ has 0 as annihilator, i.e. $S(0, x) = S(x, 0) = 0$ for any $x \in [0, 1]$. A semi-copula $S$ is a triangular norm (t-norm for short) if it is commutative and associative. The aggregation functions $T_M$ and $T_D$ given by $T_M(x, y) = \min(x, y)$ and $T_D(x, y) = \min(x, y)$ whenever $\max(x, y) = 1$, and $T_D(x, y) = 0$ elsewhere, are examples of t-norms. Moreover, for any semi-copula $S$ the inequality $T_D \leq S \leq T_M$ holds. A semi-copula $S$ is a quasi-copula if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0, 1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

$$|S(x', y') - S(x, y)| \leq |x' - x| + |y' - y|.$$

A semi-copula $S$ is a copula if it is 2-increasing, i.e. for any $x, x', y, y' \in [0, 1]$ such that $x \leq x'$ and $y \leq y'$, it holds that $V_S([x, x'] \times [y, y']) := S(x', y) + S(x, y) - S(x', y') - S(x, y') \geq 0$.

$V_S$ is called the volume of the rectangle $[x, x'] \times [y, y']$. Any copula is a quasi-copula since the 2-increasingness of a semi-copula implies its 1-Lipschitz continuity. The (quasi-)copulas $T_M$ and $T_L$ with $T_L(x, y) = \max(x + y - 1, 0)$, are respectively the greatest and the smallest (quasi-)copulas, i.e. for any (quasi-)copula $C$, it holds that $T_L \leq C \leq T_M$.

The diagonal section of a $[0, 1]^2 \rightarrow [0, 1]$ function $F$ is the function $\delta_F : [0, 1] \rightarrow [0, 1]$ defined by $\delta_F(x) = F(x, x)$. A diagonal function [13] is a function $\delta : [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;
(D2) $\delta$ is increasing;
(D3) for all $x \in [0, 1]$, it holds that $\delta(x) \leq x$;
(D4) $\delta$ is 2-Lipschitz continuous, i.e. for all $x, x' \in [0, 1]$ it holds that

$$|\delta(x') - \delta(x)| \leq 2|x' - x|.$$

The set of all diagonal functions will be denoted by $D$. The set of all $[0, 1] \rightarrow [0, 1]$ functions that satisfy conditions D1 and D2 (resp. D1, D2 and D3) will be denoted by $D_A$ (resp. $D_S$). The diagonal section of a copula $C$ is a diagonal function. Conversely, for any diagonal function $\delta$, there exists at least one copula $C$ with diagonal section $\delta_C = \delta$. For instance, the copula

$$C_S(x, y) = \min \left( x, y, \frac{\delta(x) + \delta(y)}{2} \right)$$
is the greatest symmetric copula with diagonal section \( \delta \{12, 14, 26\} \). Similarly, the opposite diagonal section of a \([0, 1]^2 \rightarrow [0, 1]\) function \( F \) is the function \( \omega_F : [0, 1] \rightarrow [0, 1] \) defined by \( \omega_F(x) = F(x, 1-x) \). An opposite diagonal function \( \omega \) is a function \( [0, 1] \rightarrow [0, 1] \) satisfying the following conditions:

**OD1** for all \( x \in [0, 1] \), it holds that \( \omega(x) \leq \min(x, 1-x) \);

**OD2** \( \omega \) is 1-Lipschitz continuous, i.e. for all \( x, x' \in [0, 1] \), it holds that

\[
|\omega(x') - \omega(x)| \leq |x' - x|.
\]

The set of all opposite diagonal functions will be denoted by \( \mathcal{O} \). The set of all \([0, 1] \rightarrow [0, 1]\) functions that satisfy condition **OD1** will be denoted by \( \mathcal{O}_S \). The opposite diagonal section \( \omega_C \) of a copula \( C \) is an opposite diagonal function. Conversely, for any opposite diagonal function \( \omega \), there exists at least one copula \( C \) with opposite diagonal section \( \omega_C = \omega \). For instance, the copula \( F_\omega \) defined by \( F_\omega(x, y) = T_L(x, y) + \min(\omega(t) \mid t \in [\min(x, 1-y), \max(x, 1-y)]) \) is the greatest copula with opposite diagonal section \( \omega \) \( [7, 23] \). Note that \( F_\omega \) is opposite symmetric \([7]\), i.e. \( F_\omega(x, y) = F_\omega(1-y, 1-x) = x+y-1 \), for any \( (x, y) \in [0, 1]^2 \). Diagonal and opposite diagonal functions have been used recently to construct several subclasses of aggregation functions such as quasi-copulas and copulas \([5, 7, 11, 12, 13, 14, 20]\).

Characteristic for the aggregation function \( T_M \) is that its surface is constituted from linear segments connecting its diagonal section to the points \((0, 1, 0)\) and \((1, 0, 0)\). Also characteristic for \( T_M \) is that its surface is constituted from linear segments connecting its opposite diagonal section to the points \((0, 0, 0)\) and \((1, 1, 1)\). Inspired by the above interpretation, we introduce a new method to construct aggregation functions. These aggregation functions are constructed by linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points \((0, 1)\) and \((1, 0)\) (resp. \((0, 0)\) and \((1, 1)\)).

This paper is organized as follows. In the next section we introduce the definition of a biconic function with a given diagonal section and characterize the class of biconic aggregation functions, as well as the classes of biconic semi-copulas, biconic quasi-copulas, biconic copulas and singular biconic copulas. The class of biconic functions with a given opposite diagonal section is introduced in Section 3. Finally, some conclusions are given.

2. Biconic functions with a given diagonal section

2.1. Biconic aggregation functions with a given diagonal section

In this subsection we introduce biconic functions with a given diagonal section. Their construction is based on linear interpolation on segments connecting the diagonal of the unit square and the points \((0, 1)\) and \((1, 0)\). Throughout this paper the convention \( \frac{0}{0} = 0 \) is assumed. For any \((x, y) \in [0, 1]^2\), we introduce in this subsection the following notations:

\[
u = \frac{x}{1+x-y}, \quad \mu = \frac{y}{1+y-x}.
\]

Let \( \delta \in \mathcal{D}_A \) and \( \alpha, \beta \in [0, 1] \). The function \( A^{\alpha, \beta}_\delta : [0, 1]^2 \rightarrow [0, 1] \) defined by

\[
A^{\alpha, \beta}_\delta(x, y) = \begin{cases} \alpha(x-y) + \frac{\delta(v)}{v}, & \text{if } y \leq x, \\ \beta(y-x) + \frac{\delta(u)}{u}, & \text{otherwise} \end{cases}
\]

(1)
is well defined. This function is called a biconic function with a given diagonal section.

**Proposition 1** Let \( \delta \in \mathcal{D}_A \). The function \( A^{\alpha, \beta}_\delta \) defined in (1) is an aggregation function if and only if

(i) the functions \( \mu_{\delta, \alpha}, \mu_{\delta, \beta} : [0, 1] \rightarrow \mathbb{R} \), defined by

\[
\mu_{\delta, \alpha}(x) = \frac{\delta(x) - \alpha}{x}, \quad \mu_{\delta, \beta}(x) = \frac{\delta(x) - \beta}{x},
\]

are increasing;

(ii) the functions \( \lambda_{\delta, \alpha}, \lambda_{\delta, \beta} : [0, 1] \rightarrow \mathbb{R} \), defined by

\[
\lambda_{\delta, \alpha}(x) = \frac{\delta(x) - \alpha}{1-x}, \quad \lambda_{\delta, \beta}(x) = \frac{\delta(x) - \beta}{1-x},
\]

are increasing.

Inspired by the above proposition, the biconic function \( A^{\alpha, \beta}_\delta \) is called a biconic aggregation function with a given diagonal section.

**Example 1** Consider the diagonal section \( \delta_M \) of \( T_M \), i.e. \( \delta_M(x) = x \) for any \( x \in [0, 1] \). Obviously, conditions (i) and (ii) of Proposition 1 are satisfied. The resulting biconic aggregation function is a Choquet integral \([4, 10]\), i.e.

\[
A^{\alpha, \beta}_{\delta_M}(x, y) = \begin{cases} \alpha x + (1-\alpha)y, & \text{if } y \leq x, \\ (1-\beta)x + \beta y, & \text{otherwise}. \end{cases}
\]

Taking \( \beta = 1-\alpha \), the resulting biconic aggregation function is the weighted arithmetic mean, i.e.

\[
A^{\alpha, 1-\alpha}_{\delta_M}(x, y) = \alpha x + (1-\alpha)y.
\]

**Proposition 2** Let \( A^{\alpha, \beta}_\delta \) be a biconic aggregation function. Then the inequality

\[
\max(\alpha x, \beta y) \leq \delta(x) \leq \min(\alpha(1-\alpha)x, \beta(1-\beta)x),
\]

(2)
holds for any \( x \in [0, 1] \).
Now we identify the functions in $\mathcal{D}_A$ which characterize the extreme biconic aggregation functions with fixed $\alpha$ and $\beta$. Let $\alpha, \beta \in [0, 1]$ and consider the functions $\delta_{\alpha, \beta}, \overline{\delta}_{\alpha, \beta} : [0, 1] \rightarrow [0, 1]$, defined by

$$
\delta_{\alpha, \beta}(x) = \begin{cases} 
\max(\alpha x, \beta x), & \text{if } x < 1, \\
1, & \text{if } x = 1, \\
\min(\alpha + (1 - \alpha)x, \beta + (1 - \beta)x), & \text{if } x > 0, \\
0, & \text{if } x = 0.
\end{cases}
$$

and

$$
\overline{\delta}_{\alpha, \beta}(x) = \begin{cases} 
\min(\alpha x, \beta x), & \text{if } x < 1, \\
1, & \text{if } x = 1, \\
\max(\alpha + (1 - \alpha)x, \beta + (1 - \beta)x), & \text{if } x > 0, \\
0, & \text{if } x = 0.
\end{cases}
$$

Obviously, $\delta_{\alpha, \beta}, \overline{\delta}_{\alpha, \beta} \in \mathcal{D}_A$ and the conditions of Proposition 1 are satisfied. Note also that for any two biconic aggregation functions $A_{\alpha_1}^{\alpha_2, \beta}$ and $A_{\beta_1}^{\alpha_2, \beta_2}$, it holds that $A_{\alpha_1}^{\alpha_2, \beta} \leq A_{\beta_1}^{\alpha_2, \beta}$ if and only if $\delta_1 \leq \delta_2$. The following proposition is then obvious.

**Proposition 3** Let $A_{\alpha, \beta}^{\delta_1}$ be a biconic aggregation function. Then it holds that

$$
A_{\alpha_1}^{\delta_1, \beta} \leq A_{\alpha_2}^{\delta_2, \beta} \leq A_{\alpha_1}^{\delta_2, \beta}.
$$

**Example 2** The functions $\delta_{0, 0}^{\alpha, \beta}$ and $\overline{\delta}_{0, 0}^{\alpha, \beta}$ are given by

$$
\delta_{0, 0}^{\alpha, \beta}(x) = \begin{cases} 
0, & \text{if } x < 1, \\
1, & \text{if } x = 1
\end{cases}
$$

and

$$
\overline{\delta}_{0, 0}^{\alpha, \beta}(x) = \begin{cases} 
1, & \text{if } x < 1, \\
0, & \text{if } x = 0.
\end{cases}
$$

The corresponding biconic aggregation functions are respectively the smallest $t$-norm, i.e. $A_{\alpha, \beta}^{0, 0} = T_D$, and the greatest $t$-norm, i.e. $A_{\alpha, \beta}^{0, 0} = T_M$.

**Example 3** The functions $\delta_{1,1}^{\alpha, \beta}$ and $\overline{\delta}_{1,1}^{\alpha, \beta}$ are given by

$$
\delta_{1,1}^{\alpha, \beta}(x) = \begin{cases} 
1, & \text{if } x > 0, \\
0, & \text{if } x = 0.
\end{cases}
$$

The corresponding biconic aggregation functions are respectively the smallest $t$-conorm, i.e. $A_{\alpha, \beta}^{1, 1} = \max(x, y)$, and the greatest $t$-conorm, i.e. $A_{\alpha, \beta}^{1, 1} = \max(x, y)$ whenever $\min(x, y) = 0$, and $A_{\alpha, \beta}^{1, 1} = 1$ elsewhere.

**Example 4** The functions $\delta_{1,0}^{\alpha, \beta}$, $\overline{\delta}_{1,0}^{\alpha, \beta}$, $\delta_{0,1}^{\alpha, \beta}$ and $\overline{\delta}_{0,1}^{\alpha, \beta}$ all coincide with $\delta_M$. The corresponding biconic aggregation functions coincide with the projection to the first and second coordinate $[2, 3]$, i.e. $A_{\alpha, \beta}^{1, 0} = A_{\alpha, \beta}^{1, 0} = x$ and $A_{\alpha, \beta}^{0, 1} = A_{\alpha, \beta}^{0, 1} = y$.

Evidently, any biconic function $A_{\alpha, \beta}^{\delta_1}$ is continuous if and only if $\delta$ is continuous. The functions $\overline{\delta}_{\alpha, \beta}$ and $\delta_{\alpha, \beta}$ need not to be continuous in general. In fact, the only case in which they are both continuous is when

$$
\max(\alpha, \beta) = 1 \text{ and } \min(\alpha, \beta) = 0.
$$

However, as Example 4 shows, it then holds that

$$
\overline{\delta}_{\alpha, \beta} = \delta = \overline{\delta}_{\alpha, \beta} = \delta_M,
$$

and $A_{\alpha, \beta}^{\delta_1}$ coincides with one of the projections.

**Proposition 4** Let $\delta \in \mathcal{D}_A$. The function $A_{\alpha, \beta}^{\delta_1}$ defined in (1) has

(i) is symmetric, i.e. $A_{\alpha, \beta}^{\delta_1}(x, y) = A_{\alpha, \beta}^{\delta_1}(y, x)$ for any $(x, y) \in [0, 1]^2$ if and only if $\alpha = \beta$;

(ii) has 0 as annihilator if and only if $\alpha = \beta = 0$;

(iii) has 1 as neutral element if and only if $\alpha = \beta = 0$.

From here on, we will only consider biconic functions with a given diagonal section that have 1 as neutral element, i.e. $\alpha = \beta = 0$. We then abbreviate $A_{0,0}^{\delta}$ as $A_{\delta}$. In this case, $A_{\delta}$ is symmetric and is given by

$$
A_{\delta}(x, y) = \begin{cases} 
y \frac{\delta(v)}{v}, & \text{if } y \leq x, \\
x \frac{\delta(u)}{u}, & \text{otherwise}.
\end{cases}
$$

Suppose that the diagonal section of a biconic aggregation function $A_{\delta}$ contains a (linear) segment with endpoints $(x_1, \delta(x_1))$ and $(x_2, \delta(x_2))$. From the definition of a biconic function with a given diagonal section, it follows that $A_{\delta}$ is linear on the triangle $T_1 := \Delta_1((x_1, x_1), (x_2, x_2), (1, 0))$ as well as on the triangle $T_2 := \Delta_2((x_1, 0), (x_2, 0), (0, 1))$. This situation is depicted in Figure 1. For any $(x, y) \in T_1$, it holds that

$$
A_{\delta}(x, y) = ax + by + c.
$$

Figure 1: Illustration for the triangles $T_1$ and $T_2$. 
Furthermore,
\[ ax_1 + bx_1 + c = \delta(x_1) \]
\[ ax_2 + bx_2 + c = \delta(x_2) \]
\[ a + c = 0. \]

Solving this system of equations and using the symmetry of \( A \), we obtain
\[ A_\delta(x, y) = \begin{cases} \frac{rx + sy - r}{t}, & \text{if } (x, y) \in T_1, \\ \frac{sx + ry - r}{t}, & \text{if } (x, y) \in T_2, \end{cases} \tag{5} \]
where
\[ r = x_1\delta(x_2) - x_2\delta(x_1), \]
\[ s = (1 - x_1)\delta(x_2) - (1 - x_2)\delta(x_1), \]
\[ t = x_2 - x_1. \]

2.2. Biconic semi-copulas with a given diagonal section

In this subsection we characterize the set of functions in \( D_S \) for which the corresponding biconic function is a semi-copula.

**Lemma 1** Let \( \delta \in D_S \). The function \( \lambda_\delta : [0, 1] \rightarrow [0, \infty[ \), defined by \( \lambda_\delta(x) = \frac{\delta(x)}{1-x} \), is increasing;

**Proposition 5** Let \( \delta \in D_S \). Then the function \( A_\delta : [0, 1]^2 \rightarrow [0, 1] \) defined in (3) is a semi-copula if and only if the function \( \mu_\delta : [0, 1] \rightarrow [0, \infty[ \), defined by \( \mu_\delta(x) = \frac{\delta(x)}{1-x} \), is increasing.

**Example 5** Consider the diagonal functions \( \delta_M \) and \( \delta_L \) with \( \delta_M \) being the diagonal section of \( T_L \), i.e. \( \delta_M(x) = \max(2x - 1, 0) \) for any \( x \in [0, 1] \). Clearly, the functions \( \mu_M \) and \( \mu_L \), defined in Proposition 5, are increasing. The corresponding biconic semi-copulas are respectively \( T_M \) and \( T_L \).

**Example 6** Consider the diagonal function \( \delta(x) = x^{1+\theta} \) with \( \theta \in [0, 1] \). Clearly, the function \( \mu_\delta \), defined in Proposition 5, is increasing for any \( \theta \in [0, 1] \). The corresponding family of biconic semi-copulas is given by
\[ A_\theta(x, y) = \begin{cases} \frac{y^{1+\theta}}{(1+y-x)^\theta}, & \text{if } y \leq x, \\ \frac{x^{1+\theta}}{(1+x-y)^\theta}, & \text{otherwise}. \end{cases} \]

**Proposition 6** Let \( A_\delta \) be a biconic semi-copula and suppose that \( \delta(x_0) = x_0 \) for some \( x_0 \in [0, 1] \). Then it holds that \( \delta(x) = x \) for any \( x \in [x_0, 1] \).

2.3. Biconic quasi-copulas with a given diagonal section

An interesting class of aggregation functions is the class of quasi-copulas. Quasi-copulas are of increasing importance in various studies in fuzzy set theory, such as preference modelling and similarity measurement [8, 9].

Next we characterize the diagonal functions for which the corresponding biconic function is a quasi-copula.

**Lemma 2** Let \( \delta \in D \). Then it holds that
(i) the function \( \nu_\delta : [0, 1] \rightarrow [2, \infty[ \), defined by \( \nu_\delta(x) = \frac{1+\delta(x)}{x} \), is decreasing;
(ii) the function \( \phi_\delta : [0, 1/2[ \cup ]1/2, 1] \rightarrow \mathbb{R} \), defined by \( \phi_\delta(x) = \frac{\delta(x)}{1-2x} \), is increasing on the interval \([0, 1/2[\) and on the interval \([1/2, 1]\).

**Proposition 7** Let \( \delta \in D \). Then the function \( A_\delta : [0, 1]^2 \rightarrow [0, 1] \) defined in (3) is a quasi-copula if and only if
(i) the function \( \mu_\delta \), defined in Proposition 5, is increasing;
(ii) the function \( \xi_\delta : [0, 1] \rightarrow [0, 1] \), defined by \( \xi_\delta(x) = \frac{x-\delta(x)}{1-x} \), is increasing.

**Example 7** Consider the diagonal function in Example 6. Clearly, the functions \( \mu_\delta \) and \( \xi_\delta \), defined in Propositions 5 and 7, are increasing. The corresponding family of biconic semi-copulas is a family of biconic quasi-copulas.

**Example 8** Consider the diagonal function \( \delta \) defined by
\[ \delta(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{6}, \\ 2x - \frac{1}{3}, & \text{if } \frac{1}{6} \leq x \leq \frac{1}{4}, \\ \frac{2}{3}x, & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 2x - 1, & \text{otherwise}. \end{cases} \]

Clearly, the function \( \mu_\delta \), defined in Proposition 5, is increasing. Note also that the function \( \xi_\delta \), defined in Proposition 7, is not increasing. Hence, the corresponding biconic semi-copula is a proper semi-copula. Consequently, the class of biconic quasi-copulas with a given diagonal section is a proper subclass of the class of biconic semi-copulas with a given diagonal section.

**Proposition 8** Let \( A_\delta \) be a biconic quasi-copula. Then it holds that
(i) if \( \delta(x_0) = x_0 \) for some \( x_0 \in ]0, 1[ \), then \( A_\delta = T_M \);
(ii) if \( \delta(x_0) = 2x_0 - 1 \) for some \( x_0 \in [1/2, 1[ \), then \( \delta(x) = 2x - 1 \) for any \( x \in [x_0, 1] \).
2.4. Biconic copulas with a given diagonal section

Another relevant class of aggregation functions is the class of copulas. Due to Sklar’s theorem [27], copulas have received increasing attention from researchers in statistics and probability theory [19]. We denote the (linear) segment with endpoints \( x, y \in [0, 1]^2 \) as
\[
(x, y) = \{ \theta x + (1 - \theta) y \mid \theta \in [0, 1] \}.
\]

**Proposition 9** Let \( \delta \) be a piecewise linear diagonal function. Then the function \( A_5 : [0, 1]^2 \to [0, 1] \) defined in (3) is a copula if and only if \( \delta \) is convex.

**Lemma 3** Let \( C_1 \) be a biconic copula and \( m_1, m_2 \in [-\infty, 0] \) such that \( m_1 > m_2 \). Consider three points \( b_1 := (x_1, x_1), b_2 := (x_2, x_2) \) and \( b_3 := (x_3, x_3) \) such that \( 0 \leq x_1 < x_2 < x_3 \leq 1 \) and the segments \( \{(b_1, 1, 0)\}, \{(b_2, 1, 0)\} \) and \( \{(b_3, 1, 0)\} \) have slopes \( m_1, -\sqrt{m_1 m_2} \) and \( m_2 \), respectively. Then it holds that
(i) there exists a rectangle \( [x, x'] \times [y, y'] \) such that the segment connecting the points \( (x, y') \) and \( (x', y) \) is a subset of the segment \( (b_2, 1, 0) \) and the points \( (x, y) \) and \( (x', y') \) are located on the segments \( \{(b_1, 1, 0)\} \) and \( \{(b_3, 1, 0)\} \) respectively.
(ii) the point \( (x_2, \delta(x_2)) \) lies below or on the segment \( \{(x_1, \delta(x_1)), (x_3, \delta(x_3))\} \).

Lemma 3 and Proposition 9 are used to show that for any convex diagonal function, the function \( A_5 \) defined in (3) is a copula.

**Proposition 10** Let \( \delta \in \mathcal{D} \). Then the function \( A_5 : [0, 1]^2 \to [0, 1] \) defined in (3) is a copula if and only if \( \delta \) is convex.

**Example 9** Consider the diagonal function in Example 6. Clearly, \( \delta \) is convex for any \( \theta \in [0, 1] \). The corresponding family of biconic semi-copulas is a family of biconic copulas.

**Example 10** Consider the diagonal function given by \( \delta(x) = \frac{x^2}{1 + \theta(1-x)^2} \) with \( \theta \in [-1, 1] \). Clearly, \( \delta \) is convex for any \( \theta \in [-1, 1] \). The corresponding family of biconic copulas is given by
\[
C_\theta(x, y) = \begin{cases} 
\frac{y^2(1+y-x)}{(1+y-x)^2 - \theta(1-x)^2}, & \text{if } y \leq x, \\
\frac{x^2(1+x-y)}{(1+x-y)^2 - \theta(1-y)^2}, & \text{otherwise}.
\end{cases}
\]

**Example 11** Consider the diagonal function \( \delta \) given by
\[
\delta(x) = \begin{cases} 
0, & \text{if } x \leq \frac{1}{3}, \\
\frac{1}{3}(4x - 1), & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3}, \\
\frac{1}{2}x, & \text{if } \frac{2}{3} \leq x \leq \frac{4}{3}, \\
2x - 1, & \text{otherwise}.
\end{cases}
\]

Clearly, the functions \( \mu_3 \) and \( \xi_5 \), defined in Propositions 5 and 7, are increasing. Note also that \( \delta \) is not convex. Hence, \( A_5 \) is a proper biconic quasi-copula. Consequently, the class of biconic copulas with a given diagonal section is a proper subclass of the class of biconic quasi-copulas with a given diagonal section.

In the following lemma the opposite symmetry property of a biconic copula with a given diagonal section is studied.

**Proposition 11** A biconic copula \( C_5 \) is opposite symmetric if and only if the function \( f(x) = x - \delta(x) \) is symmetric with respect to the point \((1/2, 1/2)\), i.e. \( \delta(x) - \delta(1-x) = 2x - 1 \) for any \( x \in [0, 1/2] \).

Next we characterize the class of singular biconic copulas with a given diagonal section. The support of a copula \( C \) is the complement of the union of all non-degenerated open rectangles of the unit square such that the \( C \)-volume of the closed rectangle is equal to zero. A copula \( C \) is called singular if its support has Lebesgue measure zero.
Proposition 12 Let $C_\delta$ be a biconic copula. Then it holds that $C_\delta$ is singular if and only if $\delta$ is piecewise linear.

Example 12 The family of biconic copulas given in (7) is a family of singular biconic copulas.

We focus now on associative biconic copulas with a given diagonal section and conclude that the only two associative biconic copulas with a given diagonal section are $T_M$ and $T_L$. Every 1-Lipschitz t-norm is a copula, while every associative copula is a $t$-norm (the commutativity can be obtained from the continuity of a copula [22]).

Proposition 13 $T_M$ and $T_L$ are the only biconic associative copulas ($1$-Lipschitz t-norms) with a given diagonal section.

We conclude this subsection by finding the intersection between the set of biconic copulas with a given diagonal section and the set of conic copulas. Conic copulas were introduced in [21] and their construction was based on linear interpolation on segments connecting the upper boundary curve of the zero-set and the point $(1,1)$. In other words, the surface of any conic copula is constituted from its zero-set and segments connecting the upper boundary curve of its zero-set to the point $(1,1,1)$. The zero-set $Z_C$ of a copula $C$ is the inverse image of the value $0$, i.e.

$$Z_C := C^{-1}([0]) = \{(x,y) \in [0,1]^2 \mid C(x,y) = 0\}.$$  

Lemma 4 Let $\delta \in D$ and suppose that $\theta = 1 - 2^{-\frac{1}{\gamma}}$, with $\gamma \in [1,\infty]$, is the maximum value such that $\delta(\theta) = 0$. Then the biconic copula $C_\delta$ has the zero set $Z_{C_\delta}$ given by

$$Z_{C_\delta} = \{(x,y) \in [0,1]^2 \mid y \leq f_{\theta_1}(x)\},$$

where the function $f_{\theta_1} : [0,1] \rightarrow [0,1]$ is given by

$$f_{\theta_1}(x) = \begin{cases} (1 - 2^{\frac{1}{\gamma}})^{-1}x + 1 & \text{if } x \leq 1 - 2^{\frac{1}{\gamma}}, \\ (1 - 2^{\frac{1}{\gamma}})(x - 1) & \text{if } x \geq 1 - 2^{\frac{1}{\gamma}}. \end{cases}$$

(6)

Due to the above lemma and the definition of a conic copula, the following proposition is obvious.

Proposition 14 Let $C_\delta$ be a biconic copula and suppose further that $C_\delta$ is a conic copula. Then it holds that $C_\delta(x,y) = (7)$

$$\begin{cases} \max(y + (1-x)(1 - 2^{\frac{1}{\gamma}}),0) & \text{if } y \leq x, \\ \max(x + (1-y)(1 - 2^{\frac{1}{\gamma}}),0) & \text{otherwise}, \end{cases}$$

with $\gamma \in [1,\infty]$. This family of copulas was introduced in [21].

3. Biconic functions with a given opposite diagonal section

In this section we introduce biconic functions with a given opposite diagonal section. Their construction is based on linear interpolation on segments connecting the opposite diagonal of the unit square and the points $(0,0)$ and $(1,1)$. For any $(x,y) \in [0,1]^2$, we introduce in this section the following notations

$$u' = \frac{x}{x+y}, \quad v' = \frac{1-y}{2-x-y}.$$ 

Let $\omega : [0,1] \rightarrow [0,1]$ and $\alpha, \beta \in [0,1]$. The function $A_{\omega,\beta} : [0,1]^2 \rightarrow [0,1]$ defined by $A_{\omega,\beta}(x,y) = \begin{cases} \alpha(1-x-y) + \frac{\omega(u')}{u'} & \text{if } x + y \leq 1, \\ \beta(x+y-1) + (1-y)\frac{\omega(v')}{v'} & \text{otherwise}. \end{cases}$

(8)

is well defined. This function is called a biconic function with a given opposite diagonal section. Clearly, the boundary conditions of an aggregation function imply that $\alpha = 0$ and $\beta = 1$. We then abbreviate $A_{\omega,1}$ as $A_\omega$ with $A_\omega(x,y) = \begin{cases} \frac{\omega(u')}{u'} & \text{if } x + y \leq 1, \\ x + y - 1 + (1-y)\frac{\omega(v')}{v'} & \text{otherwise}. \end{cases}$

(9)

(9)

Clearly, the function $A_\omega$ defined in (9) has $1$ as neutral element. Therefore, if $A_\omega$ is an aggregation function then it is also a semi-copula.

In the next proposition, we characterize the functions in $O_S$ for which the corresponding biconic function is a biconic aggregation function.

Proposition 15 Let $\omega \in O_S$. Then the function $A_\omega : [0,1]^2 \rightarrow [0,1]$ defined in (9) is an aggregation function if and only if

(i) the functions $\mu_\omega, \rho_\omega : [0,1] \rightarrow [0,1]$, defined by $\mu_\omega(x) = \frac{x}{1-x}$, $\rho_\omega(x) = \frac{1-x}{1-x}$, are decreasing;  
(ii) the functions $\lambda_\omega, \xi_\omega : [0,1] \rightarrow [0,1]$, defined by $\lambda_\omega(x) = \frac{x}{1-x}$ and $\xi_\omega = \frac{1-x}{1-x}$, are increasing.

Let $C$ be a quasi-copula (resp. copula) with opposite diagonal section $\omega$. The function $C'$, defined by $C'(x,y) = C(x,1-y)$, is again a quasi-copula (resp. copula) whose diagonal section $C_{\delta}$ is given by $\delta(x) = x - \omega(x)$. This transformation permits to derive in a straightforward manner the conditions that have to be satisfied by an opposite diagonal function to obtain a biconic quasi-copula (resp. copula), which has that opposite diagonal function as opposite diagonal section.

Proposition 16 Let $\omega \in O$. Then the function $A_\omega : [0,1]^2 \rightarrow [0,1]$ defined in (9) is a quasi-copula if and only if the functions $\mu_\omega$ and $A_\omega$, defined in Proposition 15, are respectively decreasing and increasing.
Proposition 17 Let $\omega \in O$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (9) is a copula if and only if $\omega$ is concave.

Example 13 Consider the opposite diagonal functions $\omega_M(x) = \min(x, 1-x)$ and $\omega_L(x) = 0$. Obviously, $\omega_M$ and $\omega_L$ are concave functions. The corresponding biconic copulas are respectively $T_M$ and $T_L$.

Example 14 Consider the opposite diagonal function $\omega(x) = x(1-x)$. Obviously, $\omega$ is concave. The corresponding biconic copula is given by

$$C_{\omega}(x,y) = \begin{cases} x(1-u'), & \text{if } x + y \leq 1, \\ x - (1-y)v', & \text{otherwise}. \end{cases}$$

We focus now on the symmetry and opposite symmetry properties of biconic copulas with a given opposite diagonal section.

Proposition 18 Let $C_\omega$ be a biconic copula. Then it holds that

(i) $C_\omega$ is opposite symmetric;
(ii) $C_\omega$ is symmetric if and only if $\omega$ is symmetric with respect to the point $(1/2, 1/2)$, i.e. $\omega(x) = \omega(1-x)$ for any $x \in [0, 1/2]$.

We conclude this section by finding the intersection between the class of biconic copulas with a given opposite diagonal section and the class of biconic copulas with a given diagonal section and the class of conic copulas.

Proposition 19 Let $C$ be a biconic copula with a given opposite diagonal section and suppose further that $C$ is a biconic copula with a given diagonal section. Then it holds that $C$ is a member of the following family

$$\theta T_M + (1-\theta)T_L \quad \text{with } \theta \in [0,1].$$

Let $C_\omega$ be a biconic copula. Due to the definition of $C_\omega$, the only possible zero-sets are

$$Z_{C_\omega} = Z_{T_M} = [0, 1]^2 \setminus [0,1]^2$$

and

$$Z_{C_\omega} = Z_{T_L} = \{(x, y) \in [0,1]^2 | x + y \leq 1\}.$$ 

Recalling that every conic copula is uniquely determined by its zero-set [21], the following proposition is clear.

Proposition 20 Let $C_\omega$ be a biconic copula with given opposite diagonal section $\omega$ and suppose further that $C_\omega$ is a conic copula. Then it holds that $C_\omega = T_M$ or $C_\omega = T_L$.

4. Conclusion

We have introduced biconic aggregation functions with a given diagonal (resp. opposite diagonal) section. We have also characterized the classes of biconic semi-copulas, quasi-copulas and copulas with a given diagonal (resp. opposite diagonal) section. The t-norms $T_M$ and $T_L$ turn out to be the only 1-Lipschitz biconic t-norms with a given diagonal section. Moreover, a copula that is a biconic copula with a given diagonal section as well as with a given opposite diagonal section turns out to be a convex combination of $T_M$ and $T_L$.

References


