On the distributive equation for t-representable t-norms generated from nilpotent and strict t-norms

Michał Baczyński

Institute of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland

Abstract

Recently, in [4], we have discussed the following distributive equation of implications $\mathcal{I}(x,\mathcal{T}_1(y,z)) = \mathcal{T}_2(\mathcal{I}(x,y),\mathcal{I}(x,z))$ over t-representable t-norms, generated from strict t-norms, in interval-valued fuzzy sets theory. In this work we continue these investigations, but with the assumption that $\mathcal{T}_1$ is generated from nilpotent t-norms, while $\mathcal{T}_2$ is generated from strict t-norms. As a byproduct result we show all solutions for the following functional equation $f(\min(u_1+v_1,a),\min(u_2+v_2,a)) = f(u_1,u_2)+f(v_1,v_2)$ related to this case.

Keywords: Interval-valued fuzzy sets, intuitionistic fuzzy sets, fuzzy implication, triangular norm, distributivity equations, functional equations.

1. Introduction

Distributivity of fuzzy implications over different fuzzy logic connectives has been studied in the recent past by many authors (see [2], [26], [7], [23], [24], [6],[3]). These equations have a very important role to play in efficient inferencing in approximate reasoning, especially in fuzzy control systems. Since all the rules of an inference engine are exercised during every inference cycle, the number of rules directly affects the computational duration of the overall application. To reduce the complexity of fuzzy "IF-THEN" rules, Combs and Andrews [9] required of the following classical tautology

$$(p \wedge q) \to r = (p \to r) \vee (q \to r).$$

Subsequently, there were many discussions (see [10], [11], [16], [22]), most of them pointed out the need for a theoretical investigation required for employing such equations, as concluded by Dick and Kandel [16], "Future work on this issue will require an examination of the properties of various combinations of fuzzy unions, intersections and implications". An overview of the most important methods that reduce the complexity of different inference systems can be found in [5, Chapter 8].

Recently, in [4], we have discussed the distributive equation of implications

$$\mathcal{I}(x,\mathcal{T}_1(y,z)) = \mathcal{T}_2(\mathcal{I}(x,y),\mathcal{I}(x,z)),$$

over t-representable t-norms, generated from strict t-norms, in interval-valued fuzzy sets theory. In this work we continue these investigations, but with the assumption that $\mathcal{T}_1$ is generated from nilpotent t-norms, while $\mathcal{T}_2$ is generated from strict t-norms. In [4], as a byproduct result, we have obtained the solutions of the following functional equation:

$$f(u_1+v_1,u_2+v_2) = f(u_1,u_2)+f(v_1,v_2),$$

satisfied for all $(u_1,u_2),(v_1,v_2) \in L^\infty$, where $L^\infty = \{(u_1,u_2) \in [0,\infty]^2 \mid u_1 \geq u_2\}$ and $f\colon L^\infty \to [0,\infty]$ is an unknown function. In this article we will present all solutions of the following equation:

$$f(\min(u_1+v_1,a),\min(u_2+v_2,a)) = f(u_1,u_2)+f(v_1,v_2),$$

satisfied for all $(u_1,u_2),(v_1,v_2) \in L^a$, where $a>0$ is fixed real number, $f\colon L^a \to [0,\infty]$ is an unknown function and $L^a = \{(u_1,u_2) \in [0,a]^2 \mid u_1 \geq u_2\}$. This equation is related to the case with nilpotent and strict t-norms. Such theoretical developments connected with solutions of different functional equations can be also useful in other topics like fuzzy mathematical morphology (see [12]) or similarity measures (cf. [8]).

2. Intuitionistic and interval-valued fuzzy sets theories

Intuitionistic fuzzy sets theory introduced by Atanassov [1] assign to each element of the universe not only a membership degree, but also a non-membership degree (for the discussion connected with the proposed terminology see [17]).

Definition 2.1. An intuitionistic fuzzy set A on X is a set

$$A = \{(x,\mu_A(x),\nu_A(x)) \ : \ x \in X\},$$

where $\mu_A,\ \nu_A\colon X \to [0,1]$ are called, respectively, the membership function and the non-membership function. Moreover they satisfy the condition

$$\mu_A(x)+\nu_A(x) \leq 1, \qquad x \in X.$$

Let us define

$$L^* = \{(x_1,x_2) \in [0,1]^2 \ : \ x_1+x_2 \leq 1\},$$
$$(x_1,x_2) \leq_{L^*} (y_1,y_2) \iff x_1 \leq y_1 \wedge x_2 \geq y_2.$$

One can easily observe that $\mathcal{L}^* = (L^*, \leq_{L^*})$ is a complete lattice with units $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Moreover, an intuitionistic fuzzy set A on X can be represented by the $\mathcal{L}^*$-fuzzy set given by $A\colon X \to L^*$.

Another extension of fuzzy sets theory is interval-valued fuzzy sets theory introduced, independently, by Sambuc [25] and Gorzałczany [19], in which to each element of the universe a closed subinterval of the unit interval is assigned – it can be used as an approximation of the unknown membership degree. Let us define

$$L^I = \{(x_1, x_2) \in [0,1]^2 : x_1 \leq x_2\},$$
$$(x_1, x_2) \leq_{L^I} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1 \wedge x_2 \leq y_2.$$

In the sequel, if $x \in L^I$, then we denote it by $x = [x_1, x_2]$. One can easily observe that $\mathcal{L}^I = (L^I, \leq_{L^I})$ is also a complete lattice with units $0_{\mathcal{L}^I} = [0, 0]$ and $1_{\mathcal{L}^I} = [1, 1]$.

Definition 2.2. An interval-valued fuzzy set on X is a mapping $A\colon X \to L^I$.

It is important to notice that in [13] it is shown that intuitionistic fuzzy sets theory is equivalent, from the mathematical point of view, to interval-valued fuzzy sets theory. In fact, we can see the point $(x_1, x_2) \in L^*$ as the interval $[x_1, 1 - x_2] \in L^I$ (and vice-verse). Since we are limited in number of pages, in this article we will discuss main results in the language of interval-valued fuzzy sets, but they can be easily transformed to the intuitionistic case.

3. Basic fuzzy connectives

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but we briefly mention some of the results employed in the rest of the work.

Definition 3.1. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. An associative, commutative operation $\mathcal{T}\colon L^2 \to L$ is called a t-norm if it is increasing and $1_{\mathcal{L}}$ is the neutral element of $\mathcal{T}$.

Definition 3.2. A t-norm T on $([0,1], \leq)$ is said to be nilpotent, if it is continuous and if each $x \in (0, 1)$ is a nilpotent element of T, i.e., if there exists $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$, where

$$x_T^{[n]} := \begin{cases} x, & \text{if } n = 1, \\ T(x, x_T^{[n-1]}), & \text{if } n > 1. \end{cases}$$

Definition 3.3. A t-norm T on $([0,1], \leq)$ is said to be strict, if it is continuous and strictly monotone, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$.

The following characterizations of nilpotent and strict t-norms are well-known in the literature.

Theorem 3.4 ([20])**.** *A function $T\colon [0,1]^2 \to [0,1]$ is a nilpotent t-norm if and only if there exists a continuous, strictly decreasing function $t\colon [0,1] \to [0,\infty)$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x, y) = t^{-1}(\min(t(x) + t(y), t(0))), \quad x, y \in [0, 1].$$

Theorem 3.5 ([20])**.** *A function $T\colon [0,1]^2 \to [0,1]$ is a strict t-norm if and only if there exists a continuous, strictly decreasing function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ and $t(0) = \infty$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x, y) = t^{-1}(t(x) + t(y)), \quad x, y \in [0, 1].$$

In our article we shall consider the following special class of t-norms.

Definition 3.6 (see [14])**.** A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called t-representable if there exist t-norms T_1 and T_2 on $([0,1], \leq)$ such that $T_1 \leq T_2$ and

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)],$$

for all $[x_1, x_2], [y_1, y_2] \in L^I$.

It should be noted that not all t-norms on $\mathcal{L}^I$ are t-representable (see [14]).

One possible definition of an implication on $\mathcal{L}^I$ is based on the well-accepted notation introduced by Fodor and Roubens [18] (see also [5], [15] and [21]).

Definition 3.7. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. A function $\mathcal{I}\colon L^2 \to L$ is called a fuzzy implication on $\mathcal{L}$ if it is decreasing with respect to the first variable, increasing with respect to the second variable and fulfills the following conditions: $\mathcal{I}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = \mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = \mathcal{I}(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{I}(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$.

4. Some new results pertaining to functional equations

In this section we show one new result related to functional equations, which will be crucial in obtaining main results.

Proposition 4.1 ([3, Proposition 3.6])**.** *Fix real $a > 0$. For a function $f\colon [0,a] \to [0,\infty]$ the following statements are equivalent:*

(i) f satisfies the functional equation

$$f(\min(x + y, a)) = f(x) + f(y),$$

for all $x, y \in [0, a]$.

(ii) Either $f = 0$, or $f = \infty$, or

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{if } x > 0, \end{cases} \quad \text{for all } x \in [0, a].$$

Proposition 4.2. *Fix real $a > 0$. Let $L^a = \{(u_1, u_2) \in [0,a]^2 : u_1 \geq u_2\}$. For a function $f\colon L^a \to [0,\infty]$ the following statements are equivalent:*

(i) f satisfies the functional equation

$$f(\min(u_1+v_1,a), \min(u_2+v_2,a)) = f(u_1,u_2) + f(v_1,v_2), \tag{A}$$

for all $(u_1,u_2),(v_1,v_2) \in L^a$.

(ii) Either

$$f = 0, \tag{S1}$$

or

$$f = \infty, \tag{S2}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0, \end{cases} \tag{S3}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0, \end{cases} \tag{S4}$$

for all $(u_1,u_2) \in L^a$.

Proof. $(ii) \implies (i)$ It is a direct calculation that all the above functions satisfy (A).

$(i) \implies (ii)$ Let a function $f\colon L^a \to [0,\infty]$ satisfy equation (A) for all $(u_1,u_2),(v_1,v_2) \in L^a$. Setting $u_1 = v_1 = a$ in (A) we get

$$f(\min(a+a,a), \min(u_2+v_2,a)) = f(a,u_2)+f(a,v_2),$$

for all $u_2, v_2 \in [0,a]$. Let us denote $f_a(x) := f(a,x)$, for $x \in [0,a]$. Therefore, we get

$$f_a(\min(u_2+v_2,a)) = f_a(u_2) + f_a(v_2),$$

for all $u_2, v_2 \in [0,a]$. For this equation we can use solutions from Proposition 4.1. We have 3 possible cases for the function f_a.

1. If $f_a = 0$, then putting $u_1 = u_2 = a$ in (A) we have
$$f(\min(a+v_1,a), \min(a+v_2,a)) = f(a,a) + f(v_1,v_2),$$
for all $(v_1,v_2) \in L^a$, so
$$f(a,a) = f(a,a) + f(v_1,v_2),$$
thus $0 = f(v_1,v_2)$ for all $(v_1,v_2) \in L^a$ and we get first solution $f = 0$, i.e., (S1).
2. If $f_a(x) = \begin{cases} 0, & \text{if } x = 0 \\ \infty, & \text{if } x > 0 \end{cases}$, then putting $u_1 = a$ in (A) we have
$$f(a, \min(u_2+v_2,a)) = f(a,u_2) + f(v_1,v_2),$$
for all $(v_1,v_2) \in L^a$. If we take $u_2 = v_2 = 0$ above, then we get
$$f(a,0) = f(a,0) + f(v_1,0), \qquad v_1 \in [0,a],$$
thus $f(v_1,0) = 0$ for all $v_1 \in [0,a]$.
If we take $u_2 = 0$ and $v_2 > 0$ above, then we get
$$f(a,v_2) = f(a,0) + f(v_1,v_2),$$
for all $v_1 \in [0,a]$, therefore $\infty = 0 + f(v_1,v_2)$, i.e., $f(v_1,v_2) = \infty$. In summary, we get the solution (S3).

Therefore, we need to solve our equation with the last possible assumption that $f_a = \infty$. Setting now $u_2 = v_2 = 0$ in (A) we get

$$f(\min(u_1+v_1,a),0) = f(u_1,0) + f(v_1,0),$$

where $u_1, v_1 \in [0,a]$. Let us denote $f^0(x) := f(x,0)$, for all $x \in [0,a]$. Hence, we obtain the following functional equation

$$f^0(\min(u_1+v_1,a)) = f^0(u_1) + f^0(v_1),$$

satisfied for all $u_1, v_1 \in [0,a]$. For this equation we again can use solutions described in Proposition 4.1. We have 3 possible cases for the function f^0.

1. If $f^0 = 0$, then $f(a,0) = 0$, which contradicts our assumption $f_a = \infty$.
2. If $f^0 = \infty$, then putting $u_1 = u_2 = 0$ in (A) we have
$$f(v_1,v_2) = f(0,0) + f(v_1,v_2),$$
for all $(v_1,v_2) \in L^a$, thus $f(v_1,v_2) = \infty$, hence we get next possible solution $f = \infty$, i.e., (S2).
3. If $f^0(x) = \begin{cases} 0, & \text{if } x = 0 \\ \infty, & \text{if } x > 0 \end{cases}$, then putting $u_2 = 0$ in (A) we have
$$f(\min(u_1+v_1,a),v_2) = f(u_1,0) + f(v_1,v_2).$$
Let us assume that $u_1 > 0$ and $v_1 = v_2$ above. Then we get
$$f(\min(u_1+v_2,a),v_2) = \infty + f(v_2,v_2),$$
hence
$$f(\min(u_1+v_2,a),v_2) = \infty,$$
for all $u_1 \in (0,a], v_2 \in [0,a]$. Observe that $\min(u_1+v_2,a) \in (v_2,a]$ and $v_2 \in [0,a]$, thus we have obtained the result that $f(x_1,x_2) = \infty$ for any $(x_1,x_2) \in L^a$ such that $x_1 > x_2$.
Let us take now $u_2 = u_1$ and $v_2 = v_1$ in (A). Then we have
$$f(\min(u_1+v_1,a), \min(u_1+v_1,a)) = f(u_1,u_1) + f(v_1,v_1),$$
for all $u_1, v_1 \in [0,a]$. Let us denote by $g(x) := f(x,x)$, for $x \in [0,a]$. Therefore we obtain the following functional equation
$$g(\min(u_1+v_1,a)) = g(u_1) + g(v_1),$$
satisfied for all $u_1, v_1 \in [0,a]$. For this equation we again can use solutions described in Proposition 4.1. We have 3 possible cases for the function g.

(a) If $g = 0$, then $f(a,a) = 0$, which contradicts our assumption $f_a = \infty$.
(b) If $g = \infty$, then $f(0,0) = \infty$, which contradicts our assumption 3. on function f^0.
(c) If $g(x) = \begin{cases} 0, & \text{if } x = 0 \\ \infty, & \text{if } x > 0 \end{cases}$, then taking into account previous calculations we get the solution (S4) in this case.

□

5. Distributive equation for t-representable t-norms

In this section we will show how we can use solutions presented in Proposition 4.2 to obtain all solutions, in particular fuzzy implications, of our main distributive equation

$$\mathcal{I}(x, \mathcal{T}_1(y,z)) = \mathcal{T}_2(\mathcal{I}(x,y), \mathcal{I}(x,z)), \qquad \text{(D1)}$$

satisfied for all $x, y, z \in L^I$, where $\mathcal{I}$ is an unknown function, $\mathcal{T}_1$ is a t-representable t-norm on $\mathcal{L}^I$ generated from nilpotent t-norms T_1, T_2 and $\mathcal{T}_2$ is a t-representable t-norm on $\mathcal{L}^I$ generated from strict t-norms T_3, T_4.

Assume that projection mappings on $\mathcal{L}^I$ are defined as the following:

$$pr_1([x_1, x_2]) = x_1, \qquad pr_2([x_1, x_2]) = x_2,$$

for $[x_1, x_2] \in L^I$. In [4] we have shown that if $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\mathcal{L}^I$ are t-representable, then

$$\begin{aligned} &g^1_{[x_1,x_2]}([T_1(y_1,z_1), T_2(y_2,z_2)]) \\ &\quad = T_3(g^1_{[x_1,x_2]}([y_1,y_2]), g^1_{[x_1,x_2]}([z_1,z_2])), \\ &g^2_{(x_1,x_2)}([T_1(y_1,z_1), T_2(y_2,z_2)]) \\ &\quad = T_4(g^2_{[x_1,x_2]}([y_1,y_2]), g^2_{[x_1,x_2]}([z_1,z_2])), \end{aligned}$$

where $[x_1, x_2] \in L^I$ is arbitrarily fixed and functions $g^1_{[x_1,x_2]}, g^2_{[x_1,x_2]} \colon L^I \to L^I$ are defined by

$$\begin{aligned} g^1_{[x_1,x_2]}(\cdot) &:= pr_1 \circ \mathcal{I}([x_1,x_2], \cdot), \\ g^2_{[x_1,x_2]}(\cdot) &:= pr_2 \circ \mathcal{I}([x_1,x_2], \cdot). \end{aligned}$$

Let us assume that $T_1 = T_2$ is a nilpotent t-norm generated from additive generator t_1 and $T_3 = T_4$ is a strict t-norm generated from additive generator t_3. Using the representations of nilpotent t-norms (Theorem 3.4) and strict t-norms (Theorem 3.5) we can transform our problem to the following equation (for a simplicity we deal only with g^1 now):

$$\begin{aligned} &g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1), t_1(0))), \\ &\qquad t_1^{-1}(\min(t_1(y_2)+t_1(z_2), t_1(0)))]) \\ &\qquad = t_3^{-1}(t_3(g^1_{[x_1,x_2]}([y_1,y_2])) \\ &\qquad\qquad + t_3(g^1_{[x_1,x_2]}([z_1,z_2]))). \end{aligned}$$

Hence

$$\begin{aligned} &t_3 \circ g^1_{[x_1,x_2]}([t_1^{-1}(\min(t_1(y_1)+t_1(z_1), t_1(0))), \\ &\qquad t_1^{-1}(\min(t_1(y_2)+t_1(z_2), t_1(0)))]) \\ &\qquad = t_3 \circ g^1_{[x_1,x_2]}([y_1,y_2]) \\ &\qquad\quad + t_3 \circ g^1_{[x_1,x_2]}([z_1,z_2]). \end{aligned}$$

Let us put $t_1(y_1) = u_1$, $t_1(y_2) = u_2$, $t_1(z_1) = v_1$ and $t_1(z_2) = v_2$. Of course $u_1, u_2, v_1, v_2 \in [0, t_1(0)]$. Moreover $[y_1, y_2], [z_1, z_2] \in L^I$, thus $y_1 \le y_2$ and $z_1 \le z_2$. The generator t_1 is strictly decreasing, so $u_1 \ge u_2$ and $v_1 \ge v_2$. If we put

$$f_{[x_1,x_2]}(u,v) := t_3 \circ pr_1 \circ \mathcal{I}([x_1,x_2], [t_1^{-1}(u), t_1^{-1}(v)]),$$

where $u, v \in [0, t_1(0)]$, $u \ge v$, then we get the following functional equation

$$\begin{aligned} &f_{[x_1,x_2]}(\min(u_1+v_1, t_1(0)), \min(u_2+v_2, t_1(0))) \\ &\qquad = f_{[x_1,x_2]}(u_1,u_2) + f_{[x_1,x_2]}(v_1,v_2), \end{aligned} \qquad (1)$$

satisfied for all $(u_1,u_2), (v_1,v_2) \in L^{t_1(0)}$. Of course function $f_{[x_1,x_2]} \colon L^{t_1(0)} \to [0,\infty]$ is unknown above. In a same way we can repeat all the above calculations, but for the function g^2, to obtain the following functional equation

$$\begin{aligned} &f^{[x_1,x_2]}(\min(u_1+v_1, t_1(0)), \min(u_2+v_2, t_1(0))) \\ &\qquad = f^{[x_1,x_2]}(u_1,u_2) + f^{[x_1,x_2]}(v_1,v_2), \end{aligned} \qquad (2)$$

satisfied for all $(u_1,u_2), (v_1,v_2) \in L^{t_1(0)}$, where

$$f^{[x_1,x_2]}(u,v) := t_3 \circ pr_2 \circ \mathcal{I}([x_1,x_2], [t_1^{-1}(u), t_1^{-1}(v)])$$

is an unknown function. Observe that (1) and (2) are exactly our functional equation (A). Therefore, using solutions of Proposition 4.2, we are able to obtain the description of the vertical section $\mathcal{I}([x_1,x_2], \cdot)$ for a fixed $[x_1,x_2] \in L^I$. Since in this proposition we have 4 possible solutions, we should have 16 different solutions of (D1). Observe now that some of these solutions are not good, since the range of $\mathcal{I}$ is L^I. Now, we will check all possibilities. Let us fix arbitrarily $[x_1,x_2] \in L^I$ and consider 16 different cases:

1. $f_{[x_1,x_2]} = 0$ and $f^{[x_1,x_2]} = 0$.
   This implies that

   $$t_3 \circ pr_1 \circ \mathcal{I}([x_1,x_2], [t_1^{-1}(u_1), t_1^{-1}(u_2)]) = 0,$$

   for all $u_1, u_2 \in [0, t_1(0)]$, $u_1 \ge u_2$, thus

   $$pr_1 \circ \mathcal{I}([x_1,x_2], [y_1,y_2]) = 1, \qquad [y_1,y_2] \in L^I.$$

   Similarly we get

   $$pr_2 \circ \mathcal{I}([x_1,x_2], [y_1,y_2]) = 1, \qquad [y_1,y_2] \in L^I.$$

   In summary, we obtain the following correct solution:

   $$\mathcal{I}([x_1,x_2], [y_1,y_2]) = [1,1] = 1_{\mathcal{L}^I}.$$

2. $f_{[x_1,x_2]} = 0$ and $f^{[x_1,x_2]} = \infty$.
This implies that

$$pr_1 \circ \mathcal{I}([x_1,x_2],[y_1,y_2]) = 1, \qquad [y_1,y_2] \in L^I,$$

while

$$pr_2 \circ \mathcal{I}([x_1,x_2],[y_1,y_2]) = 0, \qquad [y_1,y_2] \in L^I.$$

In summary we get the following function

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = [1,0],$$

but this solution is not correct, since $[1,0] \notin L^I$.

3. $f_{[x_1,x_2]} = 0$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$
Similarly as above one can check that such situation does not give the correct solution.

4. $f_{[x_1,x_2]} = 0$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$
One can check that such situation does not give the correct solution.

5. $f_{[x_1,x_2]} = \infty$ and $f^{[x_1,x_2]} = 0$.
This implies that

$$pr_1 \circ \mathcal{I}([x_1,x_2],[y_1,y_2]) = 0, \qquad [y_1,y_2] \in L^I,$$

while

$$pr_2 \circ \mathcal{I}([x_1,x_2],[y_1,y_2]) = 1, \qquad [y_1,y_2] \in L^I.$$

In summary we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = [0,1].$$

6. $f_{[x_1,x_2]} = \infty$ and $f^{[x_1,x_2]} = \infty$.
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = [0,0] = 0_{\mathcal{L}^I}.$$

7. $f_{[x_1,x_2]} = \infty$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [0,1], & \text{if } y_2 = 1, \\ [0,0], & \text{if } y_2 < 1. \end{cases}$$

8. $f_{[x_1,x_2]} = \infty$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [0,1], & \text{if } y_1 = 1, \\ [0,0], & \text{if } y_1 < 1. \end{cases}$$

9. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$ and $f^{[x_1,x_2]} = 0$.
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [1,1], & \text{if } y_2 = 1, \\ [0,1], & \text{if } y_2 < 1. \end{cases}$$

10. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$ and $f^{[x_1,x_2]} = \infty$.
One can check that such situation does not give the correct solution.

11. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$.
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [1,1], & \text{if } y_2 = 1, \\ [0,0], & \text{if } y_2 < 1. \end{cases}$$

12. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$.
One can check that such situation does not give the correct solution. Indeed, when $[y_1,y_2] = [0.5,1]$, then we get

$$\mathcal{I}([x_1,x_2],[0.5,1]) = [1,0].$$

13. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$ and $f^{[x_1,x_2]} = 0$.
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [1,1], & \text{if } y_1 = 1, \\ [0,1], & \text{if } y_1 < 1. \end{cases}$$

14. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0, \end{cases}$ and $f^{[x_1,x_2]} = \infty$.
One can check that such situation does not give the correct solution.

15. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ \infty, & \text{if } u_2 > 0. \end{cases}$.
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [1,1], & \text{if } y_1 = 1, \\ [0,1], & \text{if } y_1 < 1 \;\&\; y_2 = 1, \\ [0,0], & \text{if } y_2 < 1. \end{cases}$$

16. $f_{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$ and $f^{[x_1,x_2]}(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ \infty, & \text{if } u_1 > 0. \end{cases}$
In this case we obtain the following correct solution:

$$\mathcal{I}([x_1,x_2],[y_1,y_2]) = \begin{cases} [1,1], & \text{if } y_1 = 1, \\ [0,0], & \text{if } y_1 < 1. \end{cases}$$

Therefore, we have obtained 10 correct vertical sections in $\mathcal{L}^I$. Finally, we need to notice that it is not possible to find at least one solution $\mathcal{I}$ which is a fuzzy implication on $\mathcal{L}^I$ in the sense of Definition 3.7. The vertical sections 5), 6), 7) and 8) are not correct in this situation since we need to have

$$\mathcal{I}([x_1,x_2],[1,1]) = [1,1],$$

for all $[x_1,x_2] \in L^I$. For all other possible solutions 1), 9), 11), 13), 15) and 16) we have

$$\mathcal{I}([0,0],[0,0]) \neq [1,1],$$

so it is not possible to find vertical solution, which is correct for $[x_1,x_2] = [0,0]$.

6. Conclusion

In this article we have discussed the following distributive equation

$$\mathcal{I}(x,\mathcal{T}_1(y,z)) = \mathcal{T}_2(\mathcal{I}(x,y),\mathcal{I}(x,z)),$$

when both t-norms are t-representable and such that $\mathcal{T}_1$ is generated from nilpotent t-norms, while $\mathcal{T}_2$ is generated from strict t-norms. In our future work we will concentrate on a dual situation, when $\mathcal{T}_1$ is generated from strict t-norms and $\mathcal{T}_2$ is generated from nilpotent t-norms.

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