A Deterministic Algorithm for Min-max and Max-min Linear Fractional Programming Problems

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Abstract

In this paper, a deterministic global optimization algorithm is proposed for solving min-max and max-min linear fractional programming problem (P) which have broad applications in engineering, management science, nonlinear system, economics and so on. By utilizing equivalent problem (Q) of the (P) and two-phase linear relaxation technique, the relaxation linear programming (RLP) about the (P) is established. The proposed algorithm is convergent to the global minimum of (P) through the successive refinement of the feasible region and solutions of a series of RLP. And finally the numerical examples are given to illustrate the feasibility of the presented algorithm.

Keywords: min-max and max-min linear fractional programming, global optimization, two-phase linearization relaxation, branch and bound technique.

1. Introduction

Fractional programming is one of the most successful fields in nonlinear optimization. The optimization of min-max and max-min several linear fractional functions is a special class of optimization between fractional programming. It has attracted the interest of practitioners and researchers for at least 30 years (Refs. 1-8). During the past 10 years, interest in these problems has been especially intense. In part, this is because since its initial development it has spawn a wide variety of application, specially multi-stage stochastic shipping, cluster analysis and multi-objective bond portfolio and so on. Another reason for the strong interest in minmax linear fractional programming problem is that from a research point of view these

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problems pose significant theoretical and computational challenges. This is mainly because these problems are global optimization problems, i.e. they are known to generally possess multiple local optima that are not globally optima. Purpose of this paper is to develop an efficient and reliable algorithm for min-max and max-min several linear fractional functions over a polytope:

\[(P): \min \max \left\{ \frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right\} \quad \text{s.t. } Ax \leq b,\]

and

\[(P1): \max \min \left\{ \frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right\} \quad \text{s.t. } Ax \leq b,\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, f_j(x), g_j(x), j = 1, \ldots, p, \) are all linear affine functions. Since problem (P1) can be transformed into equivalent problem (P), therefore, in the following we will only consider solving problem (P).

The above min-max and max-min linear fractional programming problem usually arise in engineering, management science, nonlinear system, economics and applied mathematics which have been reviewed in Refs. 1 and 9. For examples, in engineering and economics, it is usually used to minimize a ratio of functions between a given period of time and a utilized resource in order to measure the efficiency or productivity of a system, in these types of problems the objective function is usually given as a min-max or max-min linear fractional functions (see Stancu-Minasian Ref. 10).

Min-max linear fractional programming problems arise in the design of electronic circuits, and also appear in the formulation of discrete and continuous rational approximation problems with respect to the Chebyshev norm, in continuous rational games, in multiobjective programming, in engineering design as well as in some portfolio selection problems discussed by Bajona-xandri and Martinezlegaz. A multi-facility location-queueing problem giving rise to problem (P) is introduced in Ref. 15. Financial planning with multiple fractional goals is discussed in Ref. 16. An application in computational geometry leading to a problem (P) can be found in Ref. 17. In case of infinitely rather than finitely many fractional problem (P) is related to fractional semi-infinite programming. Many applications in engineering give rise to such a problem (P) when a lower bound for the smallest eigenvalue of an elliptic differential operator is to be determined.

To our knowledge, many algorithms is proposed by utilizing duality theory for obtaining local optimum of problem (P). A rich dual theory in case of min-max several linear fractional functions exists and is reviewed in Ref. 9. Many approaches essentially lead to the same dual

\[\min \{ \sup_{x \in \mathbb{R}} x^T F(x) - u^T H(x) : u \geq 0, v \geq 0, v \neq 0 \}\]

where \( F = \{ f_1, \ldots, f_p \}, G = \{ g_1, \ldots, g_p \}. \) A rather direct approach to such a dual theory for problem (P) is given in Ref. 20. In Ref. 21, the authors modified the duality concepts proposed in Ref. 22. In Ref. 23, the authors use Clarke's generalized gradient to introduce three types of dual programs and derive corresponding duality relations. An extensively recent treatment of duality for (P) is given in Ref. 25.

Except for dual methods, large number of algorithms proposed by finding an optimal solution of the parametric problem

\[\pi(q) = \max_{1 \leq i \leq p} (\min_{x \in \mathbb{R}^n} f_i(x) - q g_i(x)),\]

which is the unique root \( q \) of \( \pi(q) = 0 \), are the most popular algorithms for solving problem (P). In Ref. 26, various modifications of the original algorithm have been proposed and tested. In Ref. 27, Gugat presented a version which is always super-linearly convergent. In Ref. 28, the authors extended the interior-point algorithm to solve the max-min linear fractional programming problems. Meanwhile, several interior-point algorithms have been proposed for the (P), which is convergent in polynomial time. In Ref. 29, an unified algorithm based on monotonic optimization theory is proposed for generalized linear fractional programming problem (P), which combines cutting plane and branch-and-bound techniques.

Recently, in Ref. 31, the authors show that a minimax fractional programming problem is equivalent to a minimax nonfractional parametric problem for a given parameter in complex space, and the necessary and sufficient optimality conditions of nondifferentiable minimax fractional programming problem with complex variables under generalized convexities is established. In Ref. 32, an improved SQP method is proposed for solving minimax problems, and a new method with small computational cost is proposed to avoid the Maratos effect. In addition, its global and superlinear convergence are obtained under some suitable conditions. In Ref. 33, Dinkelbach’s global optimization approach for finding the global maximum of the fractional programming problem is discussed. Using the lagrangian function definition for this type of problem, the Kuhn–Tucker saddle point and stationary-point...
problems are established. In addition, via the concepts of Mond–Weir type duality and Schaible type duality, a
general dual problem is formulated and some weak,
strong and converse duality theorems are proven. In Ref.
34, optimality conditions are proved for a class of
generalized fractional minimax programming problems
involving $B(p, r)$-invexity functions. Subsequently,
these optimality conditions are utilized as a basis for
constructing various duality models for this type of
fractional programming problems and proving
appropriate duality theorems.

In this paper, we will present a deterministic
algorithm using branch and bound technique for
globally solving problem (P). By utilizing equivalent
problem and two-phase linear relaxation technique, the
proposed algorithm is convergent to the global
minimum of the (P) through the successive refinement
of the linear relaxation of feasible region of the
objective function and solutions of a series of (RLP).
And finally numerical examples show the feasibility of
the proposed algorithm.

2. Equivalent Problem and Linearization
Technique

For solving problem (P), we first solve the following
$2n$ linear programming problems:

$$
\mathcal{X} = \left\{ \min x_i \right\}
\left\{ \begin{array}{l}
\text{s.t.} \quad Ax \leq b,
\end{array} \right.
$$

and

$$
\mathcal{X} = \left\{ \max x_i \right\}
\left\{ \begin{array}{l}
\text{s.t.} \quad Ax \leq b,
\end{array} \right.
$$

where $i = 1, 2, \ldots, n$. We can easily obtain the initial
lower bound and upper bound of each variable. Then we
derive an initial rectangle $X^0$ denoted by:

$$
X^0 = \{ x \} x_i \leq x_i \leq \bar{x}_i, i = 1, 2, \ldots, n\}.
$$

Let

$$
h_j(x) = \frac{f_j(x)}{g_j(x)}, j = 1, 2, \ldots, p,
$$

we introduce the equivalent problem (Q) of the (P) as
follows.

$$
\begin{cases}
\min t \\
\text{s.t.} \quad h_j(x) - t \leq 0, j = 1, 2, \ldots, p, \\
Ax \leq b, \\
x \in X^0.
\end{cases}
$$

Theorem 1. Problem (P) and (Q) have the same global
optimal solutions and optimal value.

In the following, we only consider solving problem (Q),
in this paper principal construction in the development
of a solution procedure for solving problem (Q) is
construction of a linear relaxation programming for
obtaining the lower bounds of the optimal value for this
problem. The concept of linear bounding functions is
introduced. For this considered problem (Q), we only
need to construct linear lower bounding function of
$h_j(x)$ in each constraint function. In this section the
developed method uses a convenient linearization
technique to derive the linear lower bounding function of
every $h_j(x), j = 1, 2, \ldots, p$.

Assumption 1. $f_j(x) > 0$ and $g_j(x) > 0$, for each
$j = 1, 2, \ldots, p$.

Certainly, if the following situation $f_j(x) < 0$ and
g_j(x) > 0$, or $f_j(x) > 0$ and $g_j(x) < 0$, hold, then the
$h_j(x)$ can be expressed as $-\frac{f_j(x)}{g_j(x)}$ or $-\frac{f_j(x)}{g_j(x)}$
which can be solved by the similar method proposed in this
paper.

In the following, for each $h_j(x)$ in constraint
function, we need to construct a linear lower bounding
function. By assumption 1, we can let $\eta_j = \ln(f_j(x))$
and $\xi_j = \ln(g_j(x))$, where $j = 1, 2, \ldots, p$. then we have
$h_j(x) = \exp(\eta_j - \xi_j)$. Here we adopt two-phase linear
relaxation method (Ref. 5). In the first-phase a linear
lower bounding function about the variables $\eta_j$ and $\xi_j$
is derived. Then in the second-phase the linear lower
bounding function about the primal variable $x$ is
constructed ultimately.

2.1. First-phase relaxation

By the convexity of the $\exp(Z)$, we can give a linear
lower bound function of the $\exp(Z)$ over the interval
$[Z', Z'']$ as follows:

$$
L(\exp(Z)) = B(1 + Z - \ln B),
$$

where

$$
B = \frac{\exp(Z') - \exp(Z'')}{Z'' - Z'},
$$

which is called tangential approximation at the point
$Z = \ln B$.

Additionally we can give a linear upper bound
function of $\exp(Z)$ over the interval $[Z', Z'']$ as
follows:

$$
U(\exp(Z)) = B(Z - Z') + \exp(Z').
$$

Based on the above discussion, for each function
$h_j(x)$ in constraints, we can construct the
corresponding first-phase lower bound function. According to the considered rectangle about the primal variable $x$ the bound about $\eta_j$ and $\xi_j$ can also be obtained. Denote by $\eta^*_j$, $\eta^_j$ the upper bound and lower bound of $\eta_j$, and by $\xi^*_j$, $\xi^_j$ the upper bound and lower bound of $\xi_j$ on the considered rectangle in the algorithm. Denote by $T^*_j, T^_j$ the lower bound and the upper bound of $\eta_j - \xi_j$ which can be derived easily.

Then we have
\[ B_j (1 + \eta_j - \xi_j - \ln B_j) \leq \exp(\eta_j - \xi_j) \]
where
\[ B_j = \frac{\exp(T^*_j) - \exp(T^_j)}{T^*_j - T^_j} . \]
Let
\[ Lh_j (\eta_j - \xi_j) = B_j (1 + \eta_j - \xi_j - \ln B_j) , \]
then the first-phase lower bound function of $h_j(x)$ about variable $x$, for some $j, j = 1, \ldots, p$, is given as follow:
\[ Lh_j (\eta_j - \xi_j) = B_j (1 + \ln(f_j(x)) - \ln(g_j(x)) - \ln B_j) . \tag{1} \]

2.2. Second-phase relaxation
Since the $\ln(Z)$ is a concave function, in the similar method as the first-phase relaxation we can obtain a linear lower bound function of the $\ln(Z)$ over the interval $[Z^*, Z^*]$ as follows:
\[ L(\ln(Z)) = C(Z - Z^*) + \ln Z^* \]
where
\[ C = \frac{\ln Z^* - \ln Z^*}{Z^* - Z^*} , \]
which is called tangential approximation at the point
\[ Z = \frac{Z^* - Z^*}{\ln Z^* - \ln Z^*} . \]

Additionally we can give a linear upper bound function of $\ln(Z)$ over the interval $[Z^*, Z^*]$ as follows:
\[ U(\ln(Z)) = CZ - 1 - \ln C . \]
And $L(\ln(Z))$, $U(\ln(Z))$ and $\ln(Z)$ satisfies the following inequality:
\[ C(Z - Z^*) + \ln Z^* \leq \ln(Z) \leq CZ - 1 - \ln C . \]
Therefore, by the above discussion we can obtain the following inequality:
\[ C_j (f_j(x) - \exp(\eta_j^*)) + \eta_j^* \leq \ln(f_j(x)) \]
\[ \leq C_j f_j(x) - 1 - \ln C_j , \]
\[ D_j (g_j(x) - \exp(\xi_j^*)) + \xi_j^* \leq \ln(g_j(x)) \]
\[ \leq D_j g_j(x) - 1 - \ln D_j , \tag{3} \]
where
\[ C_j = \frac{\eta^*_j - \eta^*_j}{\exp(\eta^*_j) - \exp(\eta^*_j)} , \]
\[ D_j = \frac{\xi^*_j - \xi^*_j}{\exp(\xi^*_j) - \exp(\xi^*_j)} . \]
by the above inequality (1) (2) and (3), then finally we derive the linear lower bounding function $L_j(x)$ of $h_j(x)$ for some $j = 1, 2, \ldots, p$, which underestimates the value of the considered function $h_j(x)$ as follows:
\[ L_j(x) = B_j [2 + C_j (f_j(x) - \exp(\eta_j^*)) + \eta_j^* - D_j (g_j(x) + \ln D_j) - \ln B_j] . \]
Obviously, $L_j(x) \leq h_j(x)$, for $\forall x \in X^k \subseteq X^0$.

2.3. Relaxation linear programming
Therefore, for $\forall X^k \subseteq X^0$, we can construct the relaxation linear programming (RLP) of the (Q) in $X^k$ as follows:
\[ \text{(RLP)} : \begin{cases} \min & t \\ \text{s.t.} & L_j(x) - t \leq 0, j = 1, 2, \ldots, p, \\ & Ax \leq b, x \in X^k . \end{cases} \]

Theorem 2. Let $\omega_j = T^*_j - T^*_j, u_j = \frac{\eta^*_j}{\eta^*_j}, v_j = \frac{\xi^*_j}{\xi^*_j}, t_j = \omega_j$, then the error $\Theta_j = h_j(x) - L_j(x) \to 0$ as $\omega_j \to 0, j = 1, 2, \ldots, p$.

Proof. Similarly as the proof of Lemma 1 in Ref. 5, let $\Theta_j = [h_j(x) - Lh_j(x)] + [Lh_j(x) - L_j(x)] = \Theta_{j1} + \Theta_{j2}$, then, first, we consider the difference $\Theta_{j1}$. It follows
\[ \Theta_{j1} = h_j(x) - Lh_j(x) = \exp(\eta_j - \xi_j) - B_j (1 + \eta_j - \xi_j - \ln B_j) . \]
Since $\Theta_{j1}$ is a convex function about $(\eta_j - \xi_j)$, for any $(\eta_j - \xi_j) \in [T^*_j, T^*_j]$ defined in former. Then it follows that $\Theta_{j1}$ can obtain the maximum $\Theta_{j1}^{\max}$ at the points $T^*_j$ or $T^*_j$. Let
\[ z_j = \frac{\exp(\omega_j) - 1}{\omega_j} , \]
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137
Algorithm for Fractional Programming

then through computing we can derive the following form: 
\[ \Theta_{j1}^{\text{max}} = \Theta_{j1}(T_j) = \Theta_{j1}(T_j^+) = \exp(T_j^+(1-z_j + z_j \ln z_j)). \]
Since \( z_j \to 1 \), as \( \omega_j \to 0 \). So it is obvious that \( \Theta_{j1}^{\text{max}} \to 0 \) as \( \omega_j \to 0 \).

Secondly, similarly we consider the difference \( \Theta_{j2} \),
it follows that 
\[ \Theta_{j2} = Bh_j(x) - L_j(x) \]
\[ = B_j (\ln(f_j(x)) - C_j(f_j(x) - \exp(\eta_j^+))) \]
\[ + B_j (D_j g_j(x) - 1 - \ln D_j - \ln(g_j(x))). \]
Let 
\[ \Theta_{j2,1} = \ln(f_j(x)) - C_j(f_j(x) - \exp(\eta_j^+)) - \eta_j^+, \]
\[ \Theta_{j2,2} = D_j g_j(x) - 1 - \ln(D_j) - \ln(g_j(x)), \]
Then 
\[ \Theta_{j2} = B_j \Theta_{j2,1} + B_j \Theta_{j2,2}. \]
Since \( \Theta_{j2,1} \) is a concave function about \( f_j(x) \), we can know \( \Theta_{j2,1} \) can attain the maximum \( \Theta_{j2,1}^{\text{max}} \) at the point \( f_j(x) = \ln(1/C_j) \). Then through computing we derive:
\[ \Theta_{j2,1}^{\text{max}} = \frac{\ln v_j}{v_j - 1} - 1 - \frac{\ln v_j}{v_j - 1}. \]
Also, since \( \Theta_{j2,2} \) is a convex function, it follows that it can attain the maximum \( \Theta_{j2,2}^{\text{max}} \) at the point \( \xi_j^+ \) or \( \xi_j^- \).
Then through computing we derive
\[ \Theta_{j2,2}^{\text{max}} = \frac{\ln u_j}{u_j - 1} - 1 - \frac{\ln u_j}{u_j - 1}. \]
Since \( v_j \to 1 \) and \( u_j \to 1 \) as \( \omega_j \to 0 \), then we have 
\[ \Theta_{j2,1}^{\text{max}} \to 0 \] and \( \Theta_{j2,2}^{\text{max}} \to 0 \) as \( \omega_j \to 0 \). Therefore from (4) we can follow that \( \Theta_{j2} \to 0 \) as \( \omega_j \to 0 \). By the above discussion it is obvious that the conclusion is followed.

The above theorem ensures each \( L_j(x) \) will approximate the corresponding function \( h_j(x) \) as \( \omega_j \to 0, j=1,\ldots,p \). Obviously, we have the optimal value of the RLP and (Q) satisfy \( v(\text{RLP}) \leq v(Q), \) i.e. the RLP provides the lower bound for the optimal value of the equivalent problem (Q). Based on the above construction method of the linear relaxation, for \( \forall X^i \subseteq X^0 \), problem RLP\( (X^i) \) provides a valid lower bound for the optimal value of problem \( Q(X^i) \).

3. Algorithm and Its convergence

In this section, a deterministic global optimization algorithm is developed to solve the (Q) based on the former linear relaxation technique. This algorithm needs to solve a sequence of relaxation linear programming over partitioned subsets of \( X^0 \) in order to find a global optimum solution. Furthermore, in order to ensure convergence to a global optimum, some bound tightening strategies can be applied to enhance the solution procedure.

The branch and bound approach is based on partitioning the set \( X^0 \) into sub-hyper-rectangles, each concerned with a node of the branch and bound tree, and each node is associated with a relaxation linear sub-problem in each sub-hyper-rectangle. Hence, at any stage \( k \) of the algorithm, suppose that we have a collection of active nodes denoted by \( \Omega_k \), say, each associated with a hyper-rectangle \( X \subseteq X^0 \), \( X \in \Omega_k \).
For each such node \( X \), we will have computed a lower bound of the optimal value of the (Q) via the solution \( LB(X) \) of the RLP, so that the lower bound of optimal value of the (Q) on the whole initial box region \( X^0 \) at stage \( k \) is given by \( LB_k = \min \{ LB(X), \forall X \in \Omega_k \} \).
Whenever the lower bounding solution for any node sub-problem, i.e., the solution of the relaxation linear programming RLP turns out to be feasible to the (Q), we update the upper bound of incumbent solution \( UB \) if necessary. Then, the active nodes collection \( \Omega_k \) will satisfy \( LB(X) < UB, \forall X \in \Omega_k \), for each stage \( k \). We now select an active node to partition its associated hyper-rectangle into two sub-hyper-rectangles as described below, computing the lower bounds for each new node as before. Upon fathoming any non-improving nodes, we obtain a collection of active nodes for the next stage, and this process is repeated until convergence is obtained.

Branching rule:
The critical element in guaranteeing convergence to a global minimum is the choice of a suitable partitioning strategy. In our paper we choose a simple and standard bisection rule. This method is sufficient to ensure convergence since it drives all the intervals to zero for the variables that are associated with the term that yields the greatest discrepancy in the employed approximation along any infinite branch of the branch and bound tree. This branching rule is as follows.

Consider any node sub-problem identified by the hyper-rectangle \( X = [\underline{x}, \overline{x}] \subseteq X^0 \), and the selection of
the branching variable $x^i$ and the partitioning of $X^i$ is then done using the following rule.

Let 
\[ q = \arg \max_{x \in \mathbb{R}^n} \left\{ (x_i - x_j) : i, j = 1, \ldots, n \right\}, \]
and partition $X^i$ by bisection the interval $[x_i, x_j]$ into the subintervals $[x_i, (x_i + x_j)/2]$ and $[(x_i + x_j)/2, x_j]$. The basic steps of the proposed global optimization algorithm are summarized as follows. Let $LB(X^i)$ refer to the optimal objective function value of (RLP) on the sub-hyper-rectangles $X^i$ and $\hat{x} = x(X^i)$ refer to an element of corresponding argmin.

**Step 1: (Initialization)**

Initialize the convergence tolerance $\varepsilon$; the iteration counter $k = 0$; the set of active node $\Omega_k = \Omega^0$; the upper bound $UB = \infty$; the set of feasible points $F := \emptyset$.

Solve the RLP($X$) for $X = \Omega^0$, obtaining $LB_0 := LB(X^0)$ and $\hat{x} = x(X^0)$. If $\hat{x}$ is feasible to the (P), update $F$ and $UB$, if necessary. If $UB - LB_0 \leq \varepsilon$, then stop with $\hat{x}$ as the prescribed solution to problem (P). Otherwise, proceed to step 2.

**Step 2: (Updating upper bound)**

Let
\[ UB = \min_{x \in F} \left\{ f_1(x), \ldots, f_n(x), g_1(x), \ldots, g_p(x) \right\}, \]
If $F := \emptyset$, then the known best feasible solution is 
\[ \hat{x} = \arg \min_{x \in F} \left\{ f_1(x), \ldots, f_n(x), g_1(x), \ldots, g_p(x) \right\}. \]

**Step 3: (Partition)**

Partition $X^i$ to get two new sub-hyper-rectangles according to the above branching rule. Call the set of new partition rectangles as $\hat{X}$. For each $X \in \hat{X}$, calculate the lower bound $LB(X)$ and $x(X)$ by solving RLP($X$).

If $LB(X) > UB$, then let $\hat{X} := \hat{X} \setminus X$.

**Step 4: (Updating lower bound)**

If $x(X)$ satisfy $LB(X) \leq UB$ and $x(X)$ is feasible to problem (P), then update $UB$, $F$ and $\hat{x}$, if necessary, and let 
\[ \Omega_{k+1} = \Omega_k \setminus \hat{X}, \]
update lower bound 
\[ LB_{k+1} = \inf_{x \in \hat{X}} LB(X). \]

**Step 5: (Fathoming)**

Let 
\[ \Omega_{k+1} = \Omega_k \setminus \{ X : UB - LB(X) \leq \varepsilon, X \in \Omega_k \}. \]

If $\Omega_{k+1} = \emptyset$, then algorithm stop, $UB$ is the global optimal value for the (P), $\hat{x}$ is a global optimization solution of problem (P);

Otherwise, let $k := k + 1$, select $X^k$ such that $X^k = \arg \min_{x \in \Omega_k} LB(X)$, $x^k := x(X^k)$, return to step 2.

**Theorem 3.** The above algorithm either terminates finitely with the solution being optimal to the (P), or generates an infinite sequence of iteration such that along any infinite branch of the branch-and-bound tree, and accumulation point of the sequence $\{LB_k\}$ will be the global minimum of the (P).

**Proof.** In Ref. 35, Horst and Tuy point out that a sufficient condition for convergence of the algorithm is that the bounding operation must be consistent and the selection operation bound improving.

Let $LB_k$ is a computed lower bound in stage $k$ and $UB$ is the best upper bound at iteration $k$ not necessarily occurring inside the same sub-rectangle with $LB_k$. Since the employed subdivision process is the bisection, the process is exhaustive. Consequently, from Theorem 2 the following formulation holds:
\[ \lim_{k \to \infty} (UB - LB_k) = 0, \]
and then it means that the employed bounding operation is consistent.

Obviously, since the partition element where the actual lower bound is attained is selected for further partition in the immediately following iteration, the employed selection operation is called for bound improving.

In summary, the proposed algorithm are satisfied with that the bounding operation is consistent and that the selection operation is bound improving, therefore according to Theorem IV.3, in Horst and Tuy 35 the employed algorithm is convergent to the global minimum of problem (P).

4. **Numerical examples**

To verify the feasibility of the proposed algorithm, compute the following several numerical examples using C++ code in Pentium IV.

**Example 1 39**.
Algorithm for Fractional Programming

\[
\begin{align*}
\text{max} & \quad \min \left\{ \begin{array}{c}
37x_1 + 73x_2 + 13 - 63x_1 - 18x_2 + 39 \\
13x_1 + 13x_3 + 13 - 13x_1 + 26x_2 + 13
\end{array} \right. \\
\text{s.t.} & \quad 5x_1 - 3x_2 = 3, \\
& \quad 1.5 \leq x_1 \leq 3, \\
& \quad 1.5 \leq x_2 \leq 4.
\end{align*}
\]

Set \( \varepsilon = 5 \times 10^{-8} \), numerical results is given as follows: number of iteration \( s = 6 \), the maximal number of active nodes necessary \( n = 7 \), execution time in seconds \( t = 0s \), obtain the optimal value \( V^* = 1.49072061 \), the optimal solution \( x_1^* = 1.5, x_2^* = 1.5 \).

**Example 2**

\[
\begin{align*}
\text{min} & \quad \max \left\{ \begin{array}{c}
3x_1 + x_2 - 2x_1 + 0.8 - 4x_1 - 2x_2 + x_3 \\
2x_1 - x_2 + x_3 - 7x_1 + 3x_2 - x_3
\end{array} \right. \\
\text{s.t.} & \quad x_1 + x_2 - x_3 \leq 1, \\
& \quad -x_1 + x_3 - x_3 \leq -1, \\
& \quad 12x_1 + 5x_2 + 12x_3 \leq 34.8, \\
& \quad 12x_1 + 12x_2 + 7x_3 \leq 29.1, \\
& \quad -6x_1 + x_2 + x_3 \leq -4.1, \\
& \quad 1.0 \leq x_1 \leq 1.1, \\
& \quad 0.55 \leq x_2 \leq 0.65, \\
& \quad 1.35 \leq x_3 \leq 1.45.
\end{align*}
\]

Set \( \varepsilon = 5 \times 10^{-8} \), numerical results is given as follows: number of iteration \( s = 6 \), the maximal number of active nodes necessary \( n = 5 \), execution time in seconds \( t = 0s \), obtain the optimal value \( V^* = 0.572810738 \), the optimal solution \( x_1^* = 1.015678086, x_2^* = 0.590676676, x_3^* = 1.403391837 \).

**Example 3.**

\[
\begin{align*}
\text{min} & \quad \max \left\{ \begin{array}{c}
2x_1 + 2x_2 - x_3 + 0.9 - 3x_1 - x_2 + x_3 \\
x_1 - x_2 + x_3 - 8x_1 + 4x_2 - x_3
\end{array} \right. \\
\text{s.t.} & \quad x_1 + x_2 - x_3 \leq 1, \\
& \quad -x_1 + x_3 - x_3 \leq -1, \\
& \quad 12x_1 + 5x_2 + 12x_3 \leq 34.8, \\
& \quad 12x_1 + 12x_2 + 7x_3 \leq 29.1, \\
& \quad -6x_1 + x_2 + x_3 \leq -4.1, \\
& \quad 1.0 \leq x_1 \leq 1.2, \\
& \quad 0.55 \leq x_2 \leq 0.65, \\
& \quad 1.35 \leq x_3 \leq 1.45.
\end{align*}
\]

Set \( \varepsilon = 5 \times 10^{-4} \), numerical results is given as follows: number of iteration \( s = 8 \), the maximal number of active nodes necessary \( n = 8 \), execution time in seconds \( t = 0s \), obtain the optimal value \( V^* = 1.346854863 \), the optimal solution is \( x_1^* = 1.016666667, x_2^* = 0.55, x_3^* = 1.45 \).

**Example 4.**

\[
\begin{align*}
\text{min} & \quad \max \left\{ \begin{array}{c}
3x_1 + x_2 - 2x_1 + 0.8 - 4x_1 - 2x_2 + x_3 \\
2x_1 - x_2 + x_3 - 7x_1 + 3x_2 - x_3
\end{array} \right. \\
\text{s.t.} & \quad x_1 + x_2 - x_3 \leq 1, \\
& \quad -x_1 + x_3 - x_3 \leq -1, \\
& \quad 12x_1 + 5x_2 + 12x_3 \leq 34.8, \\
& \quad 12x_1 + 12x_2 + 7x_3 \leq 29.1, \\
& \quad -6x_1 + x_2 + x_3 \leq -4.1, \\
& \quad 1.0 \leq x_1 \leq 1.2, \\
& \quad 0.50 \leq x_2 \leq 0.65, 1.35 \leq x_3 \leq 1.45.
\end{align*}
\]

Set \( \varepsilon = 5 \times 10^{-4} \), the results is given as follows: number of iteration \( s = 7 \), the maximal number of active nodes necessary \( n = 8 \), execution time in seconds \( t = 0s \), obtain the optimal value \( V^* = 2.284427051 \), the optimal solution is \( x_1^* = 1.008333333, x_2^* = 0.5, x_3^* = 1.45 \).

Numerical result show that our algorithm can globally solve min-max and max-min linear fractional programming problem (P) and (P1) on a microcomputer.

**5. Conclusion**

In this paper, by utilizing equivalent transformation, two-phase linear relaxation method and branch-and-bound technique, we present a deterministic algorithm for solving min-max and max-min linear fractional programming problem (P) and (P1) which have broad applications in engineering, management science, nonlinear system, economics and so on. The proposed algorithm is convergent to the global minimum of (P) through the successive refinement of the feasible region and solutions of a series of RLP. And finally several numerical examples are given to illustrate the feasibility and efficiency of the presented algorithm.

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References