

Orthogonality of Imprecise Matrices

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Abstract

In this paper, we have studied some properties of imprecise numbers, namely, interval numbers and triangular fuzzy numbers. Also we studied the imprecise matrices of different kinds. Using the operations defined on them, we have proposed the orthogonality of such matrices and some theorem for those type of orthogonal matrices.

Keywords: Interval numbers, interval matrices, triangular fuzzy numbers, triangular fuzzy arithmetic, triangular fuzzy matrices.

1. Introduction

It is well known that the matrix formulation of a mathematical formulae, gives extra advantage to handle/ study the problem. When some problems are not solved by classical matrices, then the concept of fuzzy matrices are useful.

Fuzzy matrices were introduced for the first time by Thomason [6], who discussed the convergence of powers of fuzzy matrix. Fuzzy matrices play an important role in science and technology. Some properties on determinant and adjoint of square fuzzy matrices

are presented by Ragab [4]. Triangular fuzzy matrices were introduced by Shyamal and Pal [5] and they presented some important properties for them. Triangular fuzzy matrices (TFMs) are formed by triangular fuzzy numbers (TFNs). Like interval numbers, various ordered relations for triangular fuzzy numbers are available in literature, which was presented by Kaufmann and Gupta [1].

In this article, we introduce the notion of orthogonality of interval matrices and triangular fuzzy matrices and also we discuss some properties of them.

2. Different types of imprecise matrices

Different types of imprecise matrices are defined by several authors.

2.1. Interval matrix

Before introducing the definition of an interval matrix (IM), we shall give some information about interval number. In our daily life, we have to face some problems such as some data or numbers can not be specified precisely or accurately due to the error of the measuring technique or instru-

ments, etc. In such cases, interval number may be used.

Definition 1 Interval Number: An interval number is defined as $A = [a_L, a_R] = \{a : a_L \leq a \leq a_R\}$ where, a_L and a_R are the real numbers called the left end point and right end point respectively of the interval A .

Another way to represent an interval number in terms of midpoint and width is as follows:

$$A = \langle m(A), w(A) \rangle, \text{ where, } m(A) = \text{midpoint of } A = \frac{a_R + a_L}{2} \text{ and } w(A) = \text{half of the width of } A = \frac{a_R - a_L}{2}.$$

A crisp real number k may be considered as a degenerate interval $[k, k] = \langle k, 0 \rangle$.

Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two interval numbers. Different binary operations between A and B are defined as below.

(i) **Addition:** $A + B = [a_L + b_L, a_R + b_R]$. Alternately, in mean-width notations, if $A = \langle m_1, w_1 \rangle$ and $B = \langle m_2, w_2 \rangle$ then, $A + B = \langle m_1 + m_2, w_1 + w_2 \rangle$

(ii) **Multiplication:** The product of two interval numbers $A = [a_L, a_R]$ and $B = [b_L, b_R]$ is given by $AB = [\min\{a_L.b_L, a_R.b_L, a_L.b_R, a_R.b_R\}, \max\{a_L.b_L, a_R.b_L, a_L.b_R, a_R.b_R\}]$.

If A and B both are positive, then $AB = [a_L.b_L, a_R.b_R]$.

The negation of an interval number $A = [a_L, a_R]$ is given by $-A = [-a_R, -a_L]$. The subtraction of two interval numbers $A = [a_L, a_R]$ and $B = [b_L, b_R]$ is given by

$$A - B = [a_L - b_R, a_R - b_L].$$

Alternately, in mean-width notations, if $A = \langle m_1, w_1 \rangle$ and $B = \langle m_2, w_2 \rangle$ then $A - B = \langle m_1 - m_2, w_1 + w_2 \rangle$.

Definition 2 Interval matrix (IM). A matrix of order $n \times n$ is said to be an interval matrix if all its elements are the interval numbers.

As for classical matrices, we shall define some operations on interval matrices. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two interval matrices of same order. Then we have the following,

(i) $A \oplus B = (a_{ij} + b_{ij})$

(ii) $A \ominus B = (a_{ij} - b_{ij})$

(iii) If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times p}$, then $AB = (c_{ij})_{m \times p}$, where $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$; $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$

(iv) $A^T = (a_{ji})$ (the transpose of A)

(v) $kA = (ka_{ij})$, where k is a scalar.

We now define special types of interval matrices corresponding to special types of classical matrices.

Definition 3 (i) Pure Null IM: An interval matrix is said to be a pure null interval matrix if all its elements are zero. i.e., all its elements are $\langle 0, 0 \rangle$. This matrix is denoted by O .

(ii) **Fuzzy null IM:** An IM is said to be a fuzzy null IM if all its elements are of the form $a_{ij} = \langle 0, \epsilon \rangle$, where $\epsilon \neq 0$.

(iii) **Pure unit IM:** A square IM is said to be a pure unit IM if $a_{ii} = [1, 1] = \langle 1, 0 \rangle$ and $a_{ij} = [0, 0] = \langle 0, 0 \rangle$, $i \neq j$, for all i, j . It is denoted by I .

(iv) **Fuzzy unit IM:** A square IM is said to be a fuzzy unit IM if $a_{ii} = \langle 1, \epsilon_1 \rangle$ and $a_{ij} = \langle 0, \epsilon_2 \rangle$, for $i \neq j$ and $i, j \in 1, 2, \dots, n$ where $\epsilon_1, \epsilon_2 \neq 0$.

2.2. Triangular fuzzy number matrix

Triangular fuzzy number matrices are special kind of fuzzy matrices. We discuss about some properties of them as given in [5].

The point m , with membership grade 1, is called mean value and α, β are the left and right hand spreads of M respectively. A triangular fuzzy number (TFN) is said to be symmetric if both its spreads are equal, i.e., if $\alpha = \beta$ and is denoted by $M = \langle m, \alpha \rangle$.

Some arithmetic operations on TFNs are given below. Let $M = \langle m, \alpha, \beta \rangle$ and $N = \langle n, \gamma, \delta \rangle$ be two TFNs. Different binary operations between M and N are defined as below.

(1) *Addition:* $M + N = \langle m, \alpha, \beta \rangle + \langle n, \gamma, \delta \rangle = \langle m + n, \alpha + \gamma, \beta + \delta \rangle$.

(2) *Scaler multiplication:* Let λ be a scaler. Then $\lambda M = \langle \lambda m, \lambda \alpha, \lambda \beta \rangle$ when $\lambda \geq 0$ and $\lambda M = \langle \lambda m, -\lambda \beta, -\lambda \alpha \rangle$ when $\lambda \leq 0$. In particular $-M = \langle -m, \beta, \alpha \rangle$.

(3) *Multiplication:* In connection with the TFNs it can be shown that shape of the membership function of $M.N$ is not necessarily triangular but if the shape of M and N are small compared to their mean values m and n , then the shape of membership function is closed to be a triangle. A good approximation is given as follows.

(a) When $M \geq 0$ and $N \geq 0$ ($M \geq 0$ if $m \geq 0$), then $M.N = \langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, m\gamma + n\alpha, m\delta + n\beta \rangle$.

(b) When $M \leq 0$ and $N \geq 0$, then $M.N = \langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, n\alpha - m\delta, n\beta - m\gamma \rangle$.

(c) When $M \leq 0$ and $N \leq 0$, then $M.N = \langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, -n\beta - m\delta, -n\alpha - m\gamma \rangle$.

When the spreads are small compared with mean values the following is a better approximation.

$\langle m, \alpha, \beta \rangle . \langle n, \gamma, \delta \rangle \simeq \langle mn, m\gamma + n\alpha - \alpha\gamma, m\delta + n\beta + \beta\delta \rangle$ for $M > 0, N > 0$.

Definition 4 Triangular Fuzzy Matrix (TFM): A triangular

fuzzy matrix of order $m \times n$ is defined as $A = (a_{ij})_{m \times n}$ where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ and $\alpha_{ij}, \beta_{ij} \geq 0$.

As for classical matrices, we define the following operations on TFMs.

Let $A=(a_{ij})$ and $B=(b_{ij})$ be two TFMs of same order. Then we have the following.

(i) $A + B=(a_{ij} + b_{ij})$.

(ii) $A - B = (a_{ij} - b_{ij})$.

(iii) For $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times p}$,

$A.B = (c_{ij})_{m \times p}$, where $c_{ij} = \sum_{k=1}^n a_{ik} . b_{kj}, i = 1, 2, \dots, m; j = 1, 2, \dots, p$.

(iv) $A^T = (a_{ji})$ (The transpose of A).

(v) $kA = (ka_{ij})$, where k is a scaler.

In connection with TFMs corresponding to special classical matrices, we now define special types of TFMs. However, because of fuzziness we will have more than one type TFM corresponding to one type of classical matrix.

Definition 5 Pure unit TFM: A square TFM $A=(a_{ij})_{n \times n}$ is said to be a pure unit TFM if $a_{ii} = \langle 1, 0, 0 \rangle$ and $a_{ij} = \langle 0, 0, 0 \rangle$ for $i \neq j, i, j \in \{1, 2, \dots, n\}$. This matrix is denoted by I .

Definition 6 Fuzzy unit TFM: A square TFM $A = (a_{ij})_{n \times n}$ is said to be a fuzzy unit TFM if $a_{ii} = \langle 1, \varepsilon_1, \varepsilon_2 \rangle$ for $i = j$ and $a_{ij} = \langle 0, \varepsilon_3, \varepsilon_4 \rangle$ for $i \neq j; i, j \in \{1, 2, \dots, n\}$, where $\varepsilon_1 . \varepsilon_2 \neq 0, \varepsilon_3 . \varepsilon_4 \neq 0$.

Definition 7 Pure triangular TFM: A square TFM $A = (a_{ij})$ of order $n \times n$ is said to be a pure triangular TFM if either $a_{ij} = \langle 0, 0, 0 \rangle$ for $i > j$ or $a_{ij} = \langle 0, 0, 0 \rangle$ for $i < j$, where $i, j \in \{1, 2, \dots, n\}$.

A pure triangular TFM $A = (a_{ij})$ of order $n \times n$ is said to be pure upper triangular TFM when $a_{ij} = \langle 0, 0, 0 \rangle$ for $i > j$ and is said to be a pure lower triangular TFM if $a_{ij} = \langle 0, 0, 0 \rangle$ for $i < j$.

Definition 8 Fuzzy triangular TFM: A square TFM $A = (a_{ij})$ of order $n \times n$ is said to be a fuzzy triangular TFM if either $a_{ij} = \langle 0, \epsilon_1, \epsilon_2 \rangle$ for $i > j$ or $a_{ij} = \langle 0, \epsilon_1, \epsilon_2 \rangle$ for $i < j$; $i, j \in \{1, 2, 3, \dots, n\}$ and $\epsilon_1, \epsilon_2 \neq 0$.

Definition 9 Symmetric TFM: A square TFM $A = (a_{ij})$ of order $n \times n$ is said to be a symmetric TFM if $A = A^T$, i.e., if $a_{ij} = a_{ji}$ for all i, j , where A^T represents the transpose of A .

Definition 10 Pure skew symmetric TFM: A square TFM $A = (a_{ij})$ of order $n \times n$ is said to be pure skew symmetric if $A = -A^T$ and $a_{ii} = \langle 0, 0, 0 \rangle$, i.e., if $a_{ij} = -a_{ji}$ for all $i \neq j$ and $a_{ii} = \langle 0, 0, 0 \rangle$.

Definition 11 Fuzzy skew Symmetric TFM: A square TFM (a_{ij}) of order $n \times n$ is said to be fuzzy skew symmetric if $A = -A^T$ and $(a_{ii}) = \langle 0, \epsilon_1, \epsilon_2 \rangle$, i.e., if $(a_{ij}) = -(a_{ji})$ for all $i \neq j$ and $a_{ii} = \langle 0, \epsilon_1, \epsilon_2 \rangle$, where $\epsilon_1, \epsilon_2 \neq 0$.

3. Orthogonality of different types of imprecise matrices

Let $A = (a_{ij})_{n \times n}$ be a real square matrix of order n and I_n be the real unit matrix of n th order. Then A is called an orthogonal matrix if $A.A^T = A^T.A = I_n$ where A^T represents the transpose of A and ‘.’ denotes the multiplication of real matrices.

Now we may define different types of imprecise orthogonal matrices in similar manner.

3.1. Pure and fuzzy orthogonal interval matrices

Let B be an interval matrix of order $n \times n$. Then B is said to be orthogonal if $B.B^T = B^T.B = I_n$, where I_n denotes pure unit interval matrix of order $n \times n$ and the operation ‘.’ denotes the multiplication of interval matrices.

Theorem 1 *There exists no purely orthogonal interval matrix other than the pure unit Interval matrix.*

Let B be an interval matrix of order $n \times n$. Then if $B.B^T = I_n$ or $B^T.B = I_n$, where I_n is fuzzy unit IM of n th order then B is called a fuzzy orthogonal IM, i.e., If $B.B^T$ or $B^T.B$ is of the form $(a_{ij})_{n \times n}$ where $a_{ii} = \langle 1, \epsilon_1 \rangle$ and $a_{ij} = \langle 0, \epsilon_2 \rangle$ for all $i \neq j$; $i, j \in 1, 2 \dots n$ and where $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$.

Theorem 2 *If A and B be any two interval matrices then $\det(A)\det(B) \neq \det(AB)$.*

Lemma 1 *If a, b are two real numbers and x be an interval, then $(a + b)x = ax + bx$, if $a \geq 0, b \geq 0$ and $(a + b)x \neq ax + bx$, otherwise.*

i.e., in general, for n real numbers $a_i, i = 1, 2 \dots n$ and for an interval x ,

$$\left(\sum_{i=1}^n a_i\right)x = \sum_{i=1}^n a_i x \text{ if } a_i \geq 0 \text{ for all } i = 1, 2, \dots, n; \text{ and } \left(\sum_{i=1}^n a_i\right)x \neq \sum_{i=1}^n a_i x, \text{ otherwise.}$$

otherwise.

Theorem 3 *If A be a real square matrix, such that $AA^T = cI$, where c is a non-zero real number and I is the unit matrix of the order of A ; and x be an interval number such that $x^2 = \frac{1}{c}[1 - \epsilon, 1 + \epsilon]$, $0 \leq \epsilon \leq 1$, then $\bar{A} = xA$ is a fuzzy orthogonal interval matrix.*

Proof. Let, $A = (a_{ij})_{n \times n}$.

Then $A.A^T = (b_{ij})_{n \times n}$ where, $b_{ij} = \sum_{k=1}^n a_{ik}a_{jk}$.

Since, $A.A^T = cI$,

then $b_{ij} = \begin{cases} c, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

Thus, for a non-diagonal element b_{pq} of $A.A^T$, there exist two index sets Λ_1 and Λ_2 such that

$$b_{pq} = \sum_{k=\Lambda_1} a_{pk}a_{qk} - \sum_{k=\Lambda_2} a_{pk}a_{qk} \text{ where}$$

$$\sum_{k=\Lambda_1} a_{pk}a_{qk} = \sum_{k=\Lambda_2} a_{pk}a_{qk}.$$

Now, $\bar{A} = xA = (xa_{ij})_{n \times n} = (\bar{a}_{ij})_{n \times n}$ (say).

and $\bar{A}.\bar{A}^T = (\bar{b}_{ij})_{n \times n}$ where,

$$\bar{b}_{ij} = \sum_{k=1}^n \bar{a}_{ik}\bar{a}_{jk}.$$

Now, for a diagonal element \bar{b}_{ii} of $\bar{A}.\bar{A}^T$,

$$\bar{b}_{ii} = \sum_{k=1}^n xa_{ik}.xa_{ik} = \sum_{k=1}^n x^2a_{ik}^2 =$$

$$x^2 \sum_{k=1}^n a_{ik}^2 \text{ (using lemma 1, since } a_{ik}^2 \geq 0 \text{)}.$$

$$= x^2 \bar{b}_{ii} = \frac{1}{c}[1 - \epsilon, 1 + \epsilon].c = [1 - \epsilon, 1 + \epsilon].$$

For the non-diagonal element \bar{b}_{pq} of $\bar{A}.\bar{A}^T$

$$\bar{b}_{pq} = \sum_{k=i}^n xa_{pk}a_{qk}.xa_{pk}a_{qk} =$$

$$\sum_{k=1}^n x^2 a_{pk}a_{qk}$$

$$= \sum_{k=\Lambda_1} x^2 a_{pk}a_{qk} - \sum_{k=\Lambda_2} x^2 a_{pk}a_{qk}.$$

$$\text{Let, } \sum_{k=\Lambda_1} x^2 a_{pk}a_{qk} = [\gamma, \delta].$$

Then, $\sum_{k=\Lambda_2} x^2 a_{pk}a_{qk}$ is also equal to

$[\gamma, \delta]$, so that $\bar{b}_{pq} = [\gamma, \delta] - [\gamma, \delta] = [\gamma - \delta, \delta - \gamma] = [-\epsilon_1, \epsilon_1]$, where, $\epsilon_1 > 0$.

Hence, the theorem.

3.1.1. Orthogonal TFM

Let $A = (a_{ij})$ be a TFM of order $n \times n$, where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ is the ij th element of A , m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are the left and right spreads of a_{ij} respectively and also $\alpha_{ij}, \beta_{ij} \geq 0$. Depending upon the kind of unit matrix, orthogonal TFM can be defined in the following two ways.

Definition 12 Purely orthogonal TFM: If $A.A^T = A^T.A = I_n$ where I_n is the pure unit matrix of n th order, then A is called purely orthogonal TFM.

Theorem 4 There exists no purely orthogonal TFM other than the pure unit TFM.

Definition 13 Fuzzy orthogonal TFM: If $A.A^T = I_n$ or $A^T.A = I_n$, where I_n is a fuzzy unit TFM of n th order, then A is called a fuzzy unit orthogonal TFM. If $A.A^T$ or $A^T.A$ is of the form $(a_{ii})_{n \times n}$, where $a_{ii} = \langle 1, \epsilon_1, \epsilon_2 \rangle$ and $a_{ij} = \langle 0, \epsilon_3, \epsilon_4 \rangle$ for all $i \neq j; i, j \in \{1, 2, 3, \dots, n\}$ and where $\epsilon_1.\epsilon_2 \neq 0$ and $\epsilon_3.\epsilon_4 \neq 0$.

Theorem 5 If A is a fuzzy orthogonal TFM then, $A.A^T \neq A.A^T$, in general.

Theorem 6 Let A be fuzzy orthogonal TFM. The diagonal elements of $A.A^T$ and $A^T.A$ are of the form $\langle 1, \alpha_{ii}, \beta_{ii} \rangle$ for $i \in 1, 2, \dots, n$ and other elements are of the form $\langle 0, \gamma_{ij}, \delta_{ij} \rangle$ for $i \neq j$ and $i, j \in 1, 2, \dots, n$.

Theorem 7 If A is a fuzzy orthogonal TFM, then $\det(A.A^T)$ is of the form $\langle 1, \alpha, \beta \rangle$.

Theorem 8 If A and B be any two TFMs then $\det(A).\det(B) \neq \det(AB)$, in general.

Lemma 2 *Multiplication is not distributive over subtraction in triangular fuzzy numbers, i.e., $a.(b - c) \neq a.b - a.c$, in general, for any three triangular fuzzy numbers a, b, c .*

Theorem 9 *If A, B be two triangular fuzzy matrices then $\det(A).\det(B)$ may not be equal to $\det(A.B)$. If they are distinct, then they differ only by their spreads but the mean will be the same in both.*

Proof. To prove the theorem we use the above lemma. If $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ where a_{ij} 's and b_{ij} 's are triangular fuzzy numbers, then $\det(A) = \sum_{j=1}^n Sgn \sigma_j \prod_{i=1}^n a_{i\sigma_j(i)}$, where σ_j 's are all possible permutations over $\{1, 2, 3, \dots, n\}$.

Clearly, $\det(A)$ is the sum of n triangular fuzzy numbers. Let $P_j = Sgn \sigma_j \prod_{i=1}^n a_{i\sigma_j(i)}$. Now since $Sgn \sigma_j$ is either 1 or -1 then some of the P_j 's will be negative so that the sum $\det(A)$ contains some subtractions.

Similarly, if $B = (b_{ij})_{n \times n}$ then $\det(B) = \sum_{j=1}^n Q_j$ where $Q_j = Sgn \phi_j \prod_{i=1}^n b_{i\phi_j(i)}$, where ϕ_j runs over all permutations in $\{1, 2, 3, \dots, n\}$. Here also some Q_j 's will be negative.

Again, $A.B = \left(\sum_{k=1}^n a_{ik}.b_{kj} \right) = (C_{ij})$ (say).

Then, $\det(A.B) = \sum_{j=1}^n Sgn \psi_j \prod_{i=1}^n C_{i\psi_j(i)}$ where ψ_j runs over all permutations in $\{1, 2, 3, \dots, n\}$. Thus $\det(A.B)$ contains addition and subtraction of only n triangular fuzzy numbers. Since $\det(A) = \sum_{j=1}^n P_j$ and $\det(B) = \sum_{j=1}^n Q_j$

then $\det(A).\det(B)$ is the product of two expressions each containing n

number of triangular fuzzy numbers among which some are negative.

Thus using the above lemma the proof of theorem follows.

Theorem 10 *If A is an orthogonal TFM, then $\det(A)$ is of the form $\langle 1, \alpha, \beta \rangle$.*

4. Conclusion

In this article the orthogonality of IMs and TFMs are investigated. At present, we are trying to find out the eigenvalues and eigenvectors of TFMs and other properties.

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