Cluster lag synchronization of complex networks with nonidentical dynamical nodes via Adaptive control

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Abstract. In this paper, synchronization analysis is investigated for an complex networks with time delay. It is assumed the nodes in the network to be nonidentical. In order to achieve cluster synchronization, this paper uses an adaptive controller on each node, by utilizing the Lyapunov stability theory, a new and useful synchronization criteria are obtained, which guarantee the complex networks under control could realize the cluster lag synchronization. Finally, numerical simulation results are also presented to verify the effectiveness of the proposed synchronization scheme.

1. Introduction

As it is well known, synchronization and control in complex networks have been known for a rather long time, and many kinds of synchronization have been introduced[1-8], such as complete synchronization, lag synchronization, projective synchronization, impulsive synchronization, cluster synchronization, partial synchronization, mixed synchronization, and so on.

Cluster synchronization in community network with nonidentical nodes is studied and several sufficient conditions for synchronization are obtained analytically. In [9], complex networks with community structure and nonidentical nodes are investigated. The node dynamics of different communities in the same community network are assumed to be nonidentical. Based on Lyapunov stability theory, sufficient conditions for achieving cluster projective synchronization are derived through designing proper controllers. In [10], H_1 cluster synchronization and state estimation for complex dynamical networks with mixed time delays is investigated. A novel concept of H_1 cluster synchronization is proposed to quantify against the exogenous disturbance of the complex networks. In [11], cluster synchronization in nonlinearly coupled delayed networks of non-identical dynamic systems is investigated. In [12], complex networks with community structure and nonidentical nodes are investigated. The cluster mixed synchronization of these networks is studied by using some linear pinning control schemes. In [13], cluster synchronization under the influence of advection is considered. In [14], Liu et al. investigated cluster synchronization for networks with cooperative and competitive weight couplings. In [15], cluster synchronization is studied in adaptive complex dynamical networks with nonidentical nodes and a novel adaptive strategy is proposed to realize global cluster synchronization.

Motivated by the above discussions, this paper investigates cluster lag synchronization in the time delay coupled complex dynamical networks. For achieving the synchronization, basing on the structure of the networks and Lyapunov stability theory, this paper uses an adaptive controller on each node, some sufficient criteria are derived to ensure cluster lag synchronization.

The paper is organized as follows: The network model is introduced followed by some definitions, lemmas and hypotheses in section 2. The cluster lag synchronization of the complex coupled networks is discussed in section 3. Simulations are obtained in section 4. Finally, in section five the various conclusions are discussed.

2. Model and preliminaries

The network with delay coupling can be described by

$$\dot{x}_i(t) = f_{\phi_i}(t, x_i(t)) + c \sum_{j=1}^N b_{ij} \Gamma x_j(t-\tau), \quad i=1, 2, \dots, N \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in R^n$ is the state vector of node i , $c > 0$ is the coupling strength. $f_{\phi_i} : R^n \rightarrow R^n$ describes the local dynamics of nodes in the ϕ_i th community. For any pair of nodes i and j , if $\phi_i \neq \phi_j$, i.e., nodes i and j belong to different communities, then $f_{\phi_i} \neq f_{\phi_j}$. $\Gamma \in R^{n \times n}$ is inner-coupling matrices, $\tau > 0$ is the time delay, $B = (b_{ij})_{N \times N}$ are the weight configuration matrices. If there is a connection from node i to node j ($j \neq i$) then $b_{ij} > 0$, otherwise, $b_{ij} = b_{ji} = 0$, and the diagonal elements of matrix B are defined as

$$b_{ii} = - \sum_{j=1, j \neq i}^N b_{ij}, \quad i=1, 2, \dots, N \quad (2)$$

When the control inputs $v_i(t)$ and $u_i(t) \in R^n$ ($i=1, 2, \dots, N$) are introduced, the controlled dynamical network with respect to network (1) can be written as

$$\begin{cases} \dot{x}_i(t) = f_{\phi_i}(t, x_i(t)) + c \sum_{j=1}^N b_{ij} \Gamma x_j(t-\tau) + u_i(t), & \phi_i(t) \in \bar{J}_{\phi_i} \\ \dot{x}_i(t) = f_{\phi_i}(t, x_i(t)) + c \sum_{j=1}^N b_{ij} \Gamma x_j(t-\tau) + v_i(t), & \phi_i(t) \in J_{\phi_i} - \bar{J}_{\phi_i} \end{cases}, \quad i=1, 2, \dots, N \quad (3)$$

where J_{ϕ_i} denotes all the nodes in the ϕ_i th community and $\bar{J}_{\phi_i}$ represents the nodes in the ϕ_i th community which have direct links with the nodes in other communities.

Definition1. Let $\{C_1, C_2, \dots, C_m\}$ denote m ($2 \leq m \leq N$) communities of the networks and $\bigcup_{i=1}^m C_i = \{1, 2, \dots, N\}$. Assume that $s(t) \in R^n$ is any smooth dynamics. The error variables as $e_i(t) = x_i(t) - s(t - \tau_{\phi_i})$ for $i=1, 2, \dots, N$, the controlled complex community network (1) is said to achieve cluster lag synchronization (CLS), if $\lim_{t \rightarrow \infty} \|x_i(t) - s(t - \tau_{\phi_i})\| = 0$, $i=1, 2, \dots, N$ for any initial conditions. where $\tau_{\phi_i} > 0$ is the time delay

Lemma1. For any two vectors x and y , a matrix $Q > 0$ with compatible dimensions, one has:
 $2x^T y \leq x^T Q x + y^T Q^{-1} y$.

Assumption 1 [16] (Global Lipschitz Condition). Suppose that there exist nonnegative constants γ , for all $\forall t \in R_+$, such that for any time-varying vectors $x(t), y(t) \in R^n$,

$$\|f_{\phi_i}(x) - f_{\phi_i}(y)\| \leq \gamma \|x - y\|,$$

where $\|\cdot\|$ denotes the 2-norm throughout the paper.

3. Main results

According to system (1) and Definition1, the error dynamical system can be derived as

$$\begin{cases} \dot{e}_i(t) = f_{\phi_i}(t, x_i(t)) - f_{\phi_i}(t, s(t - \tau_{\phi_i})) + c \sum_{j=1}^N b_{ij} \Gamma e_j(t - \tau) \\ \quad + c \sum_{i=1}^N b_{ij} \Gamma s(t - \tau_{\phi_i}) + u_i(t) & \phi_i(t) \in \bar{J}_{\phi_i}, i=1, 2, \dots, N. \\ \dot{e}_i(t) = f_{\phi_i}(t, x_i(t)) - f_{\phi_i}(t, s(t - \tau_{\phi_i})) + c \sum_{j=1}^N b_{ij} \Gamma e_j(t - \tau) \\ & \phi_i(t) \in J_{\phi_i} - \bar{J}_{\phi_i} \end{cases} \quad (4)$$

According to the diffusive coupling condition (2) of the matrix A, B we have

$$c \sum_{i=1}^N b_{ij} \Gamma s(t - \tau_{\phi_i}) = 0, \quad i \in J_{\phi_i} - \bar{J}_{\phi_i} \quad (5)$$

On the basis of this property, for achieving cluster synchronization, we design the controllers as follows:

$$\begin{cases} u_i(t) = -c \sum_{i=1}^N b_{ij} \Gamma s(t - \tau_{\phi_i}) - d_i e_i(t), & i \in \bar{J}_{\phi_i} \\ v_i(t) = -d_i e_i(t), & i \in J_{\phi_i} - \bar{J}_{\phi_i} \end{cases} \quad (6)$$

where $\dot{d}_i = k_i e_i^T(t) e_i(t)$, d_i are the feedback strength and k_i are arbitrary positive constants.

Theorem 1. Suppose Assumptions 1 hold. Consider the network (1) via control law (6). If the following conditions hold:

$$\lambda_{\max}(PP^T) < \frac{2}{c}(d - \gamma) - 1.$$

Then, the system (3) is cluster lag synchronization. Where d is a sufficiently large positive constant to be determined.

Proof. Construct the following Lyapunov functional

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t) + \frac{c}{2} \int_{t-\tau}^t \sum_{i=1}^N e_i^T(\xi) e_i(\xi) d\xi + \frac{1}{2} \sum_{i=1}^N \frac{(d_i - d)^2}{k_i}$$

Calculating the derivative of $V(t)$, we have

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^N e_i^T(t) [f_{\phi_i}(t, x_i(t)) - f_{\phi_i}(t, s(t - \tau_{\phi_i})) + c \sum_{j=1}^N b_{ij} \Gamma e_j(t - \tau) - d_i e_i(t)] + \frac{c}{2} \sum_{i=1}^N e_i^T(t) e_i(t) \\ &\quad - \frac{c}{2} \sum_{i=1}^N e_i^T(t - \tau) e_i(t - \tau) + \sum_{i=1}^N (d_i - d) e_i^T(t) e_i(t) \\ &\leq \sum_{i=1}^N e_i^T(t) [f_{\phi_i}(t, x_i(t)) - f_{\phi_i}(t, s(t - \tau_{\phi_i}))] + c \sum_{i=1}^N \sum_{j=1}^N b_{ij} e_i^T(t) \Gamma e_j(t - \tau) + \frac{c}{2} \sum_{i=1}^N e_i^T(t) e_i(t) \\ &\quad - \frac{c}{2} \sum_{i=1}^N e_i^T(t - \tau) e_i(t - \tau) - \sum_{i=1}^N d e_i^T(t) e_i(t) \end{aligned}$$

By the Assumption1, we have

$$\begin{aligned} \dot{V}(t) &\leq \sum_{i=1}^N e_i^T(t) \gamma e_i(t) + c e^T(t) (B \otimes \Gamma) e(t - \tau) + \frac{c}{2} e^T(t) e(t) - \frac{c}{2} e^T(t - \tau) e(t - \tau) \\ &\quad - \sum_{i=1}^N d e_i^T(t) e_i(t) \end{aligned}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T \in R^{nN}$, $P = (B \otimes \Gamma)$, where $\otimes$ represents the Kronecker product. Then

$$\dot{V}(t) \leq c e^T(t) P e(t - \tau) + \frac{c}{2} e^T(t) e(t) - \frac{c}{2} e^T(t - \tau) e(t - \tau) + \gamma e^T(t) e(t) - d e^T(t) e(t)$$

By the Lemmal, we have

$$\begin{aligned} \dot{V}(t) &\leq \frac{c}{2} e^T(t) P P^T e(t) + \frac{c}{2} e^T(t) e(t) + \gamma e^T(t) e(t) - d e^T(t) e(t) \\ &\leq \left(\gamma + \frac{c}{2} \lambda_{\max}(P P^T) + \frac{c}{2} - d \right) e^T(t) e(t) \end{aligned}$$

Therefore, if we have $\lambda_{\max}(P P^T) < \frac{2}{c}(d - \gamma) - 1$ then $\dot{V}(t) \leq 0$. Cluster lag synchronization in the network (3) is achieved under the controllers (6), this completes the proof.

4. Illustrative examples

In this section, the nodes dynamics are the following well-known modified Chua's circuit [17] with different system parameters considering the network (3). $x_i(t) = (x_{1i}(t), x_{2i}(t), x_{3i}(t))^T$, $f_1(t, x_i(t)) = D_1 x_i(t) + h_1(x_i(t))$, $f_2(t, x_i(t)) = D_2 x_i(t) + h_2(x_i(t))$, $f_3(t, x_i(t)) = D_3 x_i(t) + h_3(x_i(t))$, $h_1(x_i) = h_2(x_i) = h_3(x_i) = (-\frac{20}{7} x_{i1}^3, 0, 0)^T$, $\Gamma = \text{Diag}\{1, 1, 1\}$ and $\tau = 0.6$, $c = 2$, $d = 15$,

$$D_1 = \begin{bmatrix} -10/7 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -12 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -10/7 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -13 & 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} -10/7 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 1 & 0 & 1 \\ 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 1 & 0 & 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{bmatrix}.$$

The following quantities are utilized to measure the process of cluster lag synchronization

$$E(t) = \sum_{i=1}^{12} \|x_i(t) - s(t - \tau_{\phi_i})\|, \quad \phi_i = 1, 2, 3 \quad (9)$$

where $E(t)$ is the error of cluster lag synchronization for this controlled network (2), cluster lag synchronization is achieved if the synchronization error $E(t)$ converges to zero as $t \rightarrow \infty$. Let $\tau_1 = 0.5, \tau_2 = 1, \tau_3 = 1.2$. Simulation results are given in Fig.1-2. Fig.1 shows the time evolution of the total synchronization errors $E(t)$. Fig.2 shows orbits of the adaptive feedback strength. The numerical results show that Theorem 1 is effective.

Fig.1. Time evolution of the synchronization errors $E(t)$.
strength d_i .

Fig.2. Orbits of the adaptive feedback

5. Conclusions

The cluster lag synchronization in community networks has been studied in this paper. According to Lyapunov stability theory, some simple and useful criteria are derived to ensure cluster lag synchronization. Particularly, the weight configuration matrix is not assumed to be symmetric and irreducible. Finally, the developed techniques are applied three complex community networks.

Numerical simulations are provided to verify the correctness and effectiveness of the derived theoretical results.

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