Relations between the solutions to matrix equations

$$AXA^* = B \text{ and } CYC^* = D$$

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Abstract: In this paper, we present some results on the relations between the solutions to matrix equations $AXA^* = B$ and $CYC^* = D$, where B and D are Hermitian matrices.

Introduction

Let $C^{m \times n}$ and $C_H^{m \times m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. For $A \in C^{m \times n}$, its range space, rank, conjugate transpose and Moore-Penrose inverse will be denoted by $R(A)$, $r(A)$, A^* and A^+ respectively. $i_+(A)$ and $i_-(A)$ denote the numbers of the positive and negative eigenvalues of a Hermitian matrix A counted with multiplicities, respectively. For convenience, we denote $E_A = I - AA^+$ and $F_A = I - A^+A$.

Consider the consistent systems

$$AXA^* = B \text{ and } CYC^* = D, \quad (1)$$

where $A, C \in C^{m \times n}$ and $B, D \in C_H^{m \times m}$, satisfying $AA^+B = B$ and $CC^+D = D$. Without loss of generality, we assume $A \neq 0$ and $C \neq 0$ in what follows. Zhang et al. [1] considered the common Hermitian least-squares solution of Eqs. 1. Liu and Yang [2] presented the common least-squares solution of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$. Zhang and Cheng [3] and Zhang [4] studied the common nonnegative-definite and positive-definite solutions to the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$.

In most of literatures, authors mainly focused on the common solution of two systems. Motivated by such situation, the aim of this article is to in-depthly investigate the relations between the solutions to consistent Eqs. 1.

For convenience, we adopt the following symbols as the solutions sets to each consistent equation in Eqs. 1,

$$S_1 = \{X \in C^{n \times n} \mid AXA^* = B\}, \quad S_2 = \{Y \in C^{n \times n} \mid CYC^* = D\}; \quad (2)$$

$$S_1^H = \{X \in C_H^{n \times n} \mid AXA^* = B\}, \quad S_2^H = \{Y \in C_H^{n \times n} \mid CYC^* = D\}. \quad (3)$$

In this article, we consider the following relations between the above sets, i.e.,

$$S_1 \cap S_2 \neq \emptyset, \quad S_1 \subseteq S_2, \quad S_1 = S_2;$$

and

$$S_1^H \cap S_2^H \neq \emptyset, \quad S_1^H \subseteq S_2^H, \quad S_1^H = S_2^H.$$

Before proceeding to the next section, we first introduce the following result which will come in handy in the proofs of our theorems.

Lemma 1.1 ([2, 5]) Let $A \in C^{m \times n}$ and $B \in C_H^{m \times m}$. Then the matrix equation $AXA^* = B$ is consistent if and only if $AA^+B = B$, in this case, a general solution can be written in the parametric form

$$X = A^+B(A^+)^* + F_A U + V F_A,$$

and the Hermitian solution is given by

$$X = A^+B(A^+)^* + F_A Z + Z^* F_A^*,$$

where $U, V, Z \in C^{n \times n}$ are arbitrary.

Lemma 1.2 ([6]) Let $A \in C^{m \times n}$, $B \in C^{m \times k}$ and $C \in C^{l \times n}$. Then

$$r\left(\begin{bmatrix} A & B \end{bmatrix}\right) = r(A) + r[(I - AA^+)B],$$

$$r\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) = r(A) + r[C(I - A^+A)].$$

Lemma 1.3 ([6]) Let $A \in C_H^{m \times m}$, $B \in C^{m \times n}$, and denote $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$. Then

$$i_{\pm}(M) = r(B) + i_{\pm}(E_B A E_B^*).$$

Lemma 1.4 ([6]) Let $A \in C_H^{m \times m}$, $B \in C^{m \times n}$ be given. Then

$$\max_{X \in C^{n \times m}} i_{\pm}[A - BX - (BX)^*] = i_{\pm}\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix},$$

$$\min_{X \in C^{n \times m}} i_{\pm}[A - BX - (BX)^*] = i_{\pm}\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix} - r(B).$$

Main results

In [2], the authors studied the common solution to a pair of matrix equations, and according to Corollary 2.1 in [2], we can obtain the following result.

Theorem 2.1 Let $A, C \in C^{m \times n}$ and $B, D \in C_H^{m \times m}$, satisfying $AA^+B = B$ and $CC^+D = D$. Then Eq. 1 have a common solution (or $S_1 \cap S_2 \neq \emptyset$) if and only if

$$r\begin{pmatrix} -B & 0 & A \\ 0 & D & C \\ A^* & C^* & 0 \end{pmatrix} = 2r\begin{pmatrix} A \\ C \end{pmatrix}.$$

Theorem 2.2 Let $A, C \in C^{m \times n}$ and $B, D \in C_H^{m \times m}$, satisfying $AA^+B = B$ and $CC^+D = D$, and S_1 and S_2 be defined by (2). Then the following statements are equivalent:

(i) $S_1 \subseteq S_2$;

$$(ii) \quad r\begin{pmatrix} -B & 0 & A \\ 0 & D & C \\ A^* & C^* & 0 \end{pmatrix} = 2r\begin{pmatrix} A \\ C \end{pmatrix} = 2r(A);$$

(iii) $CA^+B(CA^+)^* = D$ and $CF_A = 0$; or $S_1 \cap S_2 \neq \emptyset$ and $R(C^*) \subseteq R(A^*)$.

Proof. Note that statement (i) is equivalent to

$$CA^+B(CA^+)^* + CF_A UC^* + CVF_A^* C^* = D$$

for all U and V . Specially, take $U = V = 0$, we have $CA^+B(CA^+)^* = D$, and for all U and V , $CF_A UC^* + CVF_A^* C^* = 0$, means that $CF_A = 0$. So, $A^+B(A^+)^* \in S_2$, i.e., $S_1 \cap S_2 \neq \emptyset$, and $R(C^*) \subseteq R(A^*)$, then statement (ii) is obvious. $\square$

Theorem 2.3 Let $A, C \in C^{m \times n}$ and $B, D \in C_H^{m \times m}$, satisfying $AA^+B = B$ and $CC^+D = D$, and S_1 and S_2 be defined by (2). Then the following statements are equivalent:

(i) $S_1 = S_2$;

$$(ii) \quad r \begin{pmatrix} -B & 0 & A \\ 0 & D & C \\ A^* & C^* & 0 \end{pmatrix} = 2r \begin{pmatrix} A \\ C \end{pmatrix} = 2r(A) = 2r(C);$$

$$(iii) \quad S_1 \cap S_2 \neq \emptyset \quad \text{and} \quad R(C^*) = R(A^*).$$

Proof. Since $S_1 = S_2$ is equivalent to $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. Hence, by Theorem 2.2, this theorem is evident. $\square$

Since the Hermitian solution of $AXA^* = B$ can be written as $X_H = \frac{1}{2}(X + X^*)$, where X is a general solution to $AXA^* = B$, i.e., $S_i^H = \left\{ \frac{1}{2}(X + X^*), X \in S_i \right\}$, $i = 1, 2$. Therefore, for the relations for the Hermitian solution sets, the following results are obvious.

Theorem 2.4 Let $A, C \in C^{m \times n}$ and $B, D \in C_H^{m \times m}$, satisfying $AA^*B = B$ and $CC^*D = D$, S_1 and S_2 be defined by (2), S_1^H and S_2^H be defined by (3). Then

$$(i) \quad S_1 \cap S_2 \neq \emptyset \quad \text{and} \quad S_1^H \cap S_2^H \neq \emptyset \quad \text{are equivalent};$$

$$(ii) \quad S_1 \subseteq S_2 \quad \text{and} \quad S_1^H \subseteq S_2^H \quad \text{are equivalent};$$

$$(iii) \quad S_1 = S_2 \quad \text{and} \quad S_1^H = S_2^H \quad \text{are equivalent}.$$

Theorem 2.5 Let $A, C \in C^{m \times n}$ and $B, D \in C_H^{m \times m}$, satisfying $AA^*B = B$ and $CC^*D = D$, S_1^H and S_2^H be defined by (3). Then

$$\max_{X \in S_1^H, Y \in S_2^H} i_{\pm}(X - Y) = n + i_{\pm} \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} - r(A) - r(C), \quad (4)$$

$$\min_{X \in S_1^H, Y \in S_2^H} i_{\pm}(X - Y) = i_{\pm} \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} - r \begin{pmatrix} A \\ C \end{pmatrix}. \quad (5)$$

In consequence, the following hold.

(i) There exist $X \in S_1^H$ and $Y \in S_2^H$ such that $X \geq Y$ if and only if

$$i_{-} \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} = r \begin{pmatrix} A \\ C \end{pmatrix}.$$

(ii) There exist $X \in S_1^H$ and $Y \in S_2^H$ such that $X > Y$ if and only if

$$i_{+} \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} = r(A) + r(C).$$

(iii) $X \geq Y$ holds for all $X \in S_1^H$ and $Y \in S_2^H$ if and only if

$$n + i_{-} \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} = r(A) + r(C).$$

(iv) $X > Y$ holds for all $X \in S_1^H$ and $Y \in S_2^H$ if and only if

$$i_+ \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} - r \begin{pmatrix} A \\ C \end{pmatrix} = n.$$

Proof. By Lemma 1.1, $X - Y$ can be written as

$$\begin{aligned} X - Y &= A^+ B(A^+)^* + F_A Z + Z^* F_A - C^+ D(C^+)^* - F_C V - V^* F_C \\ &= A^+ B(A^+)^* - C^+ D(C^+)^* - (F_C \quad -F_A) \begin{pmatrix} V \\ Z \end{pmatrix} - (V^* \quad Z^*) \begin{pmatrix} F_C \\ -F_A \end{pmatrix}, \end{aligned}$$

where $V, Z \in C^{n \times n}$ are arbitrary. Applying Lemma 1.4, we get

$$\begin{aligned} \max_{X \in S_1^H, Y \in S_2^H} i_{\pm}(X - Y) &= i_{\pm} \begin{pmatrix} A^+ B(A^+)^* - C^+ D(C^+)^* & F_C & -F_A \\ F_C & 0 & 0 \\ -F_A & 0 & 0 \end{pmatrix}, \\ \min_{X \in S_1^H, Y \in S_2^H} i_{\pm}(X - Y) &= i_{\pm} \begin{pmatrix} A^+ B(A^+)^* - C^+ D(C^+)^* & F_C & -F_A \\ F_C & 0 & 0 \\ -F_A & 0 & 0 \end{pmatrix} - r(F_C \quad -F_A). \end{aligned}$$

Applying Lemmas 1.2-1.3 and elementary block matrix operations, we obtain

$$\begin{aligned} & i_{\pm} \begin{pmatrix} A^+ B(A^+)^* - C^+ D(C^+)^* & F_C & -F_A \\ F_C & 0 & 0 \\ -F_A & 0 & 0 \end{pmatrix} \\ &= i_{\pm} \begin{pmatrix} A^+ B(A^+)^* - C^+ D(C^+)^* & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & C^* & 0 \\ I_n & 0 & 0 & 0 & -A^* \\ 0 & C & 0 & 0 & 0 \\ 0 & 0 & -A & 0 & 0 \end{pmatrix} - r(A) - r(C) \\ &= i_{\pm} \begin{pmatrix} 0 & I_n & I_n & C^+ D & A^+ B \\ I_n & 0 & 0 & C^* & 0 \\ I_n & 0 & 0 & 0 & -A^* \\ D(C^+)^* & C & 0 & 0 & 0 \\ B(A^+)^* & 0 & -A & 0 & 0 \end{pmatrix} - r(A) - r(C) \\ &= i_{\pm} \begin{pmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C^* & A^* \\ 0 & 0 & C & -2D & 0 \\ 0 & 0 & -A & 0 & 2B \end{pmatrix} - r(A) - r(C) \\ &= n + i_{\pm} \begin{pmatrix} 0 & C^* & A^* \\ C & -2D & 0 \\ A & 0 & 2B \end{pmatrix} - r(A) - r(C), \\ & r(F_C \quad -F_A) = r(F_C \quad F_A) = r \begin{pmatrix} I_n & I_n \\ C & 0 \\ 0 & A \end{pmatrix} - r(A) - r(C) = n + r \begin{pmatrix} A \\ C \end{pmatrix} - r(A) - r(C). \end{aligned}$$

According to the above results, then (4) and (5) are evident.

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