Daily Budget Optimization Based on the Newsboy Model for Advertisers in the SSA Markets

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Abstract

This paper provides a method for planing the optimal daily budget in the online sponsored search advertising market, which is now the most important marketing place. The proposed method is mainly based the well-known newsboy model, which has been widely applied in single period inventory management settings. We model this budget optimization question as a decision problem under a situation where both the future price and demand are uncertain. We also indicate that advertisers will invest more money in the SSA market if the potential profit of clicks in this market is relatively larger than the financial cost of money.

Keywords: Newsboy, Budget, SSA

1. Introduction

Sponsored search advertising (SSA) is the most prevailing and important online advertising instrument provided by search engines. According to a secret document revealed by AdAge [1], in June this year, the top three big spenders in Google AdWords system have spent $8.08M, $6.67M, and $5.95M for advertising, respectively.

In the SSA market, advertisers submit advertisements to search engine and buy some keywords that related to their advertisements. If one of these keywords matching the query of a search engine user, the search engine will show one of these advertisements along with the standard search results. If the user clicks on the link, then the advertiser should pay a certain fee to the search engine for bringing this potential customer to its website. For more detailed information on SSA, please refer to [1, 2].

Keywords used in SSA market are sold by auctions. Two major auction formats have been used in practice: the generalized first-price and the generalized second-price mechanisms [1–4]. In this paper, we do not want to discuss the details of specific mechanisms. Readers please refer to [4–10] for the discussions of equilibrium in keywords auction mechanisms; and [11–17] for the design and comparison of new auction mechanisms.

In this paper, we propose a daily

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[1]http://tech.fortune.cnn.com/2010/09/06/adage-reveals-secret-google-adwords-big-spenders/

budget optimization method for the advertisers based on the well-known single period newsboy (or newsvendor) model [18–20]. First, we extend the classical newsboy model to address individual decision problem in a situation where both the future price and future demand are uncertain. Then, we use this extended model in the SSA market setting to obtain the optimal daily budget. Based on this model, we conclude that the advertisers would prefer to spend more money in the SSA market, if the potential profit of clicks is relatively higher than the cost of money. Otherwise, they would like to decline the money spent in this market.

The remainder of this paper is organized as follows. We illustrate how to formulate the budget optimization problem as a newsboy model in Section 2. Section 3 provides some examples and further discussions on our model. Finally, Section 4 summarizes our concluding remarks and lists some future research directions.

2. Daily Budget Optimization

In this paper, we consider a macroscope setting that an advertiser faced in the SSA market, in which we do not care details of the specific auction mechanisms used by search engines; and treat the daily budget optimization as an individual decision problem under uncertainties. However, it is also possible to model this problem as a game as have done in [21, 22], we will try this model in our future work.

2.1. Newsboy Model

Let C be the daily clicks an advertisement can received, it is a random variable defined on $[c_m, c_M]$ with density $g(c)$ and distribution $G(c)$, where c_m

and c_M are the minimum and maximum of daily clicks, respectively. Similarly, let P be the average price of a click on this advertisement, it is a random variable defined on $[p_m, p_M]$ with density $h(p)$ and distribution $H(p)$, where p_m and p_M are the minimum and maximum of price, respectively. In general, these two random variables are correlated, so we denote their joint density and distribution as $f(c, p)$ and $F(c, p)$; and let $\Omega = [c_m, c_M] \times [p_m, p_M]$ represent the domain of these two random variables.

We formulate the daily budget allocation problem as a single-period newsboy model [18–20]. To do so, let q be the daily budget an advertiser submitted. If the actual daily expenditure cp is less than the pre-submitted budget q, the unspent money $q - cp$ will cause an overage cost $\gamma_o(q - cp)$, where γ_o is the unit cost of overage. However, if the advertisers do not put enough budget into the SSA system, the search engines will shut off their advertisements. Thus, advertisers may lose some potential profit in this situation. Let γ_s be the unit potential profit of shortage, then the total shortage loss can be represented as $\gamma_s(cp - q)$. In general, γ_o can be treated as the risk-free interest, whereas γ_s is the average expected rate of ROI (return-on-investment) of the SSA market.

Using these symbols, we can denote the total loss of submitting daily budget q as

$$L(q) = \iint_{\Omega_s} \gamma_s(cp - q)f(c, p)\mathrm{d}c\mathrm{d}p$$
$$+ \iint_{\Omega_o} \gamma_o(q - cp)f(c, p)\mathrm{d}c\mathrm{d}p, \quad (1)$$

where the first term represents the total loss of shortage and the second term is the total overage loss; Ω_s is the re-

gion of shortage, whereas Ω_o is the region of shortage, such that $\Omega = \Omega_s \cup \Omega_o$. Both Ω_s and Ω_o can be empty, to reflect the situations of absolute shortage when the budget is very small, i.e., $q \leq c_m p_m$, and absolute overage when the budget is extremely high, i.e., $q \geq c_M p_M$. Figure 1 shows some simple examples.

2.2. Budget Optimization

In the following, we discuss how to find the optimal daily budget $q^\star$, i.e., the one minimizes the total loss $L(q)$ given by equation (1). To form a complete analysis, we must consider the following three situations: $c_m p_M \leq c_M p_m$, $c_m p_M = c_M p_m$, and $c_m p_M \geq c_M p_m$. However, all three situations use the same analysis method, thus, we only present the analysis of situation where $c_m p_M \leq c_M p_m$ in the rest of this section. According to figure 1, we still have to analyze five different cases.

Case 1: $q \leq c_m p_m$. Obviously, in this case, q is absolutely shortage, thus $\Omega_o = \emptyset$ and $\Omega_s = \Omega$. The total loss $L(q)$ is then

$$
\begin{aligned}
L(q) &= \iint_\Omega \gamma_s(cp - q)f(c,p)\mathrm{d}c\mathrm{d}p \\
&= \gamma_s \iint_\Omega cpf(c,p)\mathrm{d}c\mathrm{d}p - \gamma_s q.
\end{aligned}
$$

Note that $L'(q) = -\gamma_s < 0$, thus we conclude that the optimal budget in this case is $q_1^\star = c_m p_m$, and the minimal total loss is $L(q_1^\star) = \gamma_s \iint_\Omega cpf(c,p)\mathrm{d}c\mathrm{d}p - \gamma_s c_m p_m$.

Case 2: $q \geq c_M p_M$. In this case, q is absolutely overage, thus $\Omega_s = \emptyset$ and $\Omega_o = \Omega$. The total loss $L(q)$ is then

$$
\begin{aligned}
L(q) &= \iint_\Omega \gamma_o(q - cp)f(c,p)\mathrm{d}c\mathrm{d}p \\
&= \gamma_o q - \gamma_o \iint_\Omega cpf(c,p)\mathrm{d}c\mathrm{d}p.
\end{aligned}
$$

Note that $L'(q) = \gamma_o > 0$, thus we conclude that the optimal budget in this case is $q_2^\star = c_M p_M$, and the minimal total loss is $L(q_2^\star) = \gamma_o c_M p_M - \gamma_o \iint_\Omega cpf(c,p)\mathrm{d}c\mathrm{d}p$.

To discuss the rest three cases, we need the following lemma on the derivative of function defined by a double definite integral.

Lemma 2.1 *Let $k(t)$, $\alpha \leq t \leq \beta$, be a function defined by the following double definite integral*

$$
k(t) = \int_{\varphi_l(t)}^{\varphi_u(t)} \mathrm{d}x \int_{\theta_l(x,t)}^{\theta_u(x,t)} f(x,y,t)\mathrm{d}y,
$$

where both $\varphi_l(t)$ and $\varphi_u(t)$ are differentiable; both $\theta_l(x,t)$ and $\theta_u(x,t)$ are differentiable with respect to t; $f(x,y,t)$ is differentiable with respect to t and continuous on region $D = \{(x,y) \mid \varphi_l(t) \leq x \leq \varphi_u(t), \ \theta_l(x,t) \leq y \leq \theta_u(x,t)\}$. Then, $k(t)$ is differentiable and its derivative can be represented as follows

$$
\begin{aligned}
k'(t) &= \varphi_u' \int_{\theta_l(\varphi_u,t)}^{\theta_u(\varphi_u,t)} f(\varphi_u(t),y,t)\mathrm{d}y \\
&\quad - \varphi_l' \int_{\theta_l(\varphi_l,t)}^{\theta_u(\varphi_l,t)} f(\varphi_l(t),y,t)\mathrm{d}y \\
&\quad + \int_{\varphi_l}^{\varphi_u} \frac{\partial \theta_u(x,t)}{\partial t} f(x,\theta_u(x,t),t)\mathrm{d}x \\
&\quad - \int_{\varphi_l}^{\varphi_u} \frac{\partial \theta_l(x,t)}{\partial t} f(x,\theta_l(x,t),t)\mathrm{d}x \\
&\quad + \iint_D \frac{\partial f(x,y,t)}{\partial t}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

The proof of this lemma is a straightforward application of the definition of limit and the integral mean value theorem, and very cumbersome, so we omit the proof details.

Case 3: $c_M p_m \leq q \leq c_M p_M$. In this case, $\Omega_s = \{(c,p) \mid \frac{q}{p_M} \leq c \leq c_M, \ \frac{q}{c} \leq p \leq p_M\}$, see figure 1(d); and $\Omega_o = \Omega_o^{\mathrm{I}} \cup \Omega_o^{\mathrm{II}}$, where $\Omega_o^{\mathrm{I}} = \left[c_m, \frac{q}{p_M}\right] \times [p_m, p_M]$ and $\Omega_o^{\mathrm{II}} = \{(c,p) \mid \frac{q}{p_M} \leq c \leq$

(a) Critical value of q (b) $c_m p_m \leq q \leq c_m p_M$

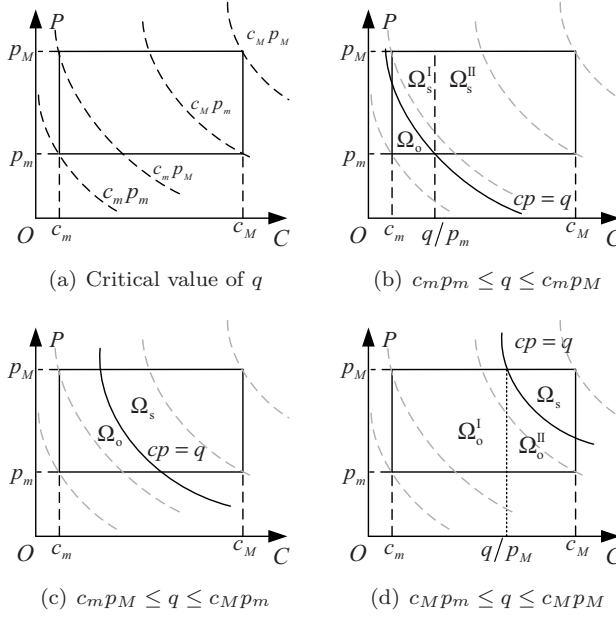(c) $c_m p_M \leq q \leq c_M p_m$ (d) $c_M p_m \leq q \leq c_M p_M$

Fig. 1: Shortage (Ω_s) and overage (Ω_o) regions in case $c_m p_M < c_M p_m$.

c_M, $p_m \leq p \leq \frac{q}{c}$}. The total loss $L(q)$ can then be represented as

$$
\begin{aligned}
L(q) &= \int_{\frac{q}{p_M}}^{c_M} \mathrm{d}c \int_{\frac{q}{c}}^{p_M} \gamma_s(cp - q)f(c,p)\mathrm{d}p \\
&+ \int_{c_m}^{\frac{q}{p_M}} \mathrm{d}c \int_{p_m}^{p_M} \gamma_o(q - cp)f(c,p)\mathrm{d}p \\
&+ \int_{\frac{q}{p_M}}^{c_M} \mathrm{d}c \int_{p_m}^{\frac{q}{c}} \gamma_o(q - cp)f(c,p)\mathrm{d}p.
\end{aligned}
$$

By using the above lemma 2.1, we obtain the derivative of the total loss $L(q)$ as follows

$$
\begin{aligned}
L'(q) &= \iint_{\Omega_o^{\mathrm{I}}} \gamma_o f(c,p)\mathrm{d}c\mathrm{d}p \\
&+ \iint_{\Omega_o^{\mathrm{II}}} \gamma_o f(c,p)\mathrm{d}c\mathrm{d}p \\
&- \iint_{\Omega_s} \gamma_s f(c,p)\mathrm{d}c\mathrm{d}p. \quad (2)
\end{aligned}
$$

To obtain the optimal daily budget $q^\star$, we simply let $L'(q^\star) = 0$, and obtain

the following condition

$$
\begin{aligned}
F\left(\frac{q^\star}{p_M}, p_M\right) &+ \int_{\frac{q^\star}{p_M}}^{c_M} \mathrm{d}c \int_{p_m}^{\frac{q^\star}{c}} f(c,p)\mathrm{d}p \\
&= \frac{\gamma_s}{\gamma_s + \gamma_o}. \quad (3)
\end{aligned}
$$

In general, the solution of the above equation is depending on the joint density function $f(c,p)$ (or joint distribution function $F(c,p)$), and does not have a explicit expression. Note that, equation (2) and (3) have the same form as those in the classical newsboy model [18–20]. We will provide further discussions on this in the next section.

Let $\mathcal{S}$ be the solution set of the above equation (3). To get the optimal budget in this case, we still need to consider the boundary value $q = c_M p_m$ and $q = c_M p_M$. Define $\hat{\mathcal{S}} = \mathcal{S} \cup \{c_M p_m, c_M p_M\}$, and the set of optimal budgets in this

case can then be represented as

$$\mathcal{Q}_1 = \left\{ q^\star \ \middle| \ q^\star = \arg\min_{q \in \hat{\mathcal{S}}} \ L(q) \right\}.$$

Case 4 and 5: Apply the same analysis to the rest two cases: $c_m p_m \leq q \leq c_m p_M$ and $c_m p_M \leq q \leq c_M p_m$, we can obtain another two candidate sets for the optimal budget: $\mathcal{Q}_2$ and $\mathcal{Q}_3$, respectively. The corresponding shortage and overage regions are indicated by figure 1(b) and 1(c). To obtain the final solution, define

$$\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3.$$

Then, the optimal budget $q^\star$ can be represented as

$$q^\star = \arg\min_{q \in \mathcal{Q}} \ L(q).$$

And the minimal total loss $L(q^\star)$ is

$$L(q^\star) = \min_{q \in \mathcal{Q}} \ L(q).$$

When there exists multiple optimums, we can safely choose the minimal one to avoid additional risk.

3. Examples and Discussions

In this section, we analysis two special cases and a numeric example to illustrate the above analysis method.

First, consider the case that random variables C and P are independent, i.e., $f(c,p) = g(c)h(p)$, or $F(c,p) = G(c)H(p)$, then equation (3) in the past section can be simplified as

$$G\left(\frac{q^\star}{p_M}\right) + \int_{\frac{q^\star}{p_M}}^{c_M} H\left(\frac{q^\star}{c}\right) g(c)\mathrm{d}c$$
$$= \frac{\gamma_s}{\gamma_s + \gamma_o}. \quad (4)$$

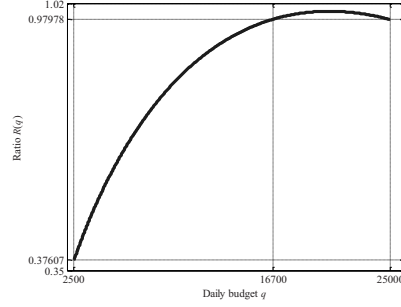Second, assume $F(c,p)$ subject to 2-dimensional uniform distribution, i.e.,

Fig. 2: $R(q)$ vs. q.

$f(c,p) \equiv \frac{1}{q_m}$, where $q_m = (c_M - c_m)(p_M - p_m)$. Obviously, in this case, random variable C and P are independent. Substitute the above density and distribution functions into equation (4), it can be simplified as

$$\frac{q^\star}{q_m}\left[1 + \ln\left(\frac{q_m}{q^\star}\right)\right] = \frac{\gamma_s}{\gamma_s + \gamma_o}. \quad (5)$$

Consider a numeric example, where $c_m = 200$, $c_M = 2500$, $p_m = 1$, and $p_M = 10$. In this example, $c_m p_M = 2000 \leq c_M p_m = 2500$. Define $R(q) = \frac{q}{q_m}\left[1 + \ln\left(\frac{q_m}{q}\right)\right]$, Figure 2 shows this function in range $2500 \leq q \leq 25000$.

Define $\beta = \frac{\gamma_s}{\gamma_s + \gamma_o}$ be the critical ratio, which reflects the relative importance of shortage and overage. As the same as the classical newsboy model [18–20], β is a increasing function with respect to the first parameter γ_s, and a decreasing function with respect to the second parameter γ_o.

Combine with figure 2, we find that there exists a threshold β_0 (about 0.3761 in the above numeric example) for the critical ratio, when $\beta < \beta_0$, the optimal budget does exist in this region. This is mainly because that this kind of advertisers are more worried about the overage cost, as a result, they put a relatively low budget into the SSA market.

When the actual critical ratio β is larger than β_0, we find that the daily budget is a increasing function with respect to β, until it reaches the maximal critical ratio $\beta = 1$. This indicates that if advertisers take more considerations to the potential profit, they will prefer to put more budget into this market.

However, as the above figure shows, there exists another key-point β_c (about 0.9798 in our example) about the critical ratio. We can find a unique optimum when the actual critical ratio is less than this critical value, whereas there exists two optimums. In this situation, we can safely choose the minimal one to avoid additional risk.

4. Conclusion

This paper provides a daily budget optimization method for the advertisers in the SSA market. The proposed method is based on the well-known single period newsboy model. We extended the classical model to be able to address individual decision problem in which both the future price and demand are uncertain, thus it can be used to obtain the optimal daily budget in a SSA market.

According to our analysis, if the advertisers were interested in obtaining potential profit, they would prefer to spend more budget in the SSA market. Otherwise, if they were worried about the financial cost of money, they would like to decline the daily budget.

In our ongoing work, we plan to extend this single period model to address dynamic budget optimization problem which requires allocating budget across a long period, such as a month, a quarter, or even a whole year. We are also dedicating to model this problem as a game theoretical, or competitive, newsboy model.

Acknowledgements

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