

## A Lexicographic Method for Matrix Games with Payoffs of Triangular Intuitionistic Fuzzy Numbers

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### Abstract

The intuitionistic fuzzy set (IF-set) has not been applied to matrix game problems yet since it was introduced by K.T.Atanassov. The aim of this paper is to develop a methodology for solving matrix games with payoffs of triangular intuitionistic fuzzy numbers (TIFNs). Firstly the concept of TIFNs and their arithmetic operations and cut sets are introduced as well as the ranking order relations. Secondly the concept of solutions for matrix games with payoffs of TIFNs is defined. A lexicographic methodology is developed to determine the solutions of matrix games with payoffs of TIFNs for both Players through solving a pair of bi-objective linear programming models derived from two new auxiliary intuitionistic fuzzy programming models. The proposed method is illustrated with a numerical example.

*Keywords:* Triangular intuitionistic fuzzy number; Intuitionistic fuzzy set; Matrix game; Mathematical programming; Lexicographic method.

### 1. Introduction

In real game situations, usually Players are not able to evaluate exactly the outcomes of game due to a lack of information. Therefore, fuzzy game theory provides an efficient framework which solves the real-life conflict problems with fuzzy information and has achieved a success. Recently, lots of papers and books<sup>1-17</sup> have

been published on this topic in which several types of fuzzy games have been investigated. However, in some situations, Players could only estimate the payoff values approximately with some imprecise degree. But it is possible that he/she is not so sure about it. In other words, there may be a hesitation about the approximate payoff values. The fuzzy set (F-set) uses only a membership function to indicate degree of belongingness to the F-set under consideration. Degree

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of non-belongingness is just automatically the complement to 1. The F-set theory is no means to incorporate the hesitation degree. In 1986 Atanassov<sup>18</sup> introduced the concept of an intuitionistic fuzzy set (IF-set), which is meant to reflect the fact that the degree of non-membership is not always equal to 1 minus the degree of membership while there may be some hesitation degree. The IF-set is characterized by two functions expressing the degree of membership and the degree of non-membership, respectively. The IF-set may express information more abundant and flexible than the F-set when uncertain information is involved. The IF-set has been applied to some areas.<sup>19-22</sup> It is essential and possible to apply the IF-set to game problems. The reason is that Players are not so sure about payoffs due to a lack of information or imprecision of the available information. In other words, Players have some degree of hesitation or uncertainty about payoffs. The IF-set can indicate Players' preference information in terms of support, opposition and neutralization. However, there exists little investigation on matrix games using the IF-set. Atanassov<sup>23</sup> firstly described a game problem using the IF-set. Dimitrov<sup>24,25</sup> applied the IF-set to some market structure problems from a view point of economic orientation. Li and Nan<sup>26</sup> studied the matrix games with payoffs of IF-sets. In this paper, we focus on reporting the solution method and computation procedure for the matrix games with payoffs of triangular intuitionistic fuzzy numbers (TIFNs). The concept of TIFNs and their arithmetic operations and cut sets are introduced as well as the ranking order relations. Matrix games with payoffs of TIFNs are formulated. Then a lexicographic methodology is developed to obtain optimal strategies for both Players through solving a pair of bi-objective linear programming models derived from new auxiliary intuitionistic fuzzy programming models.

The rest of this paper is organized as follows. Section 2 introduces the concept, arithmetic operations and cut sets of TIFNs. Furthermore, the membership degree and non-membership degree average indexes of an TIFN are defined as well as the ranking order relations between TIFNs. In Section 3 the concepts of a matrix game with payoffs of TIFNs and solutions are defined. A pair of auxiliary intuitionistic fuzzy programming models for Players and a lexicographic methodology are proposed to solve the matrix games with payoffs of TIFNs. A numerical example and

concluding remarks are given in Sections 4 and 5, respectively.

## 2. Definitions and Notations

### 2.1. The definition and operations of TIFNs

In this section, TIFNs and their operations are defined as follows.

**Definition 1.** An TIFN  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} \rangle$  is an IF-set on the real number set  $\mathbb{R}$ , whose membership function and non-membership function are defined as in Fig. 1 as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} w_{\tilde{a}}(x - \underline{a}) / (a - \underline{a}) & \text{if } \underline{a} \leq x < a \\ w_{\tilde{a}} & \text{if } x = a \\ w_{\tilde{a}}(\bar{a} - x) / (\bar{a} - a) & \text{if } a < x \leq \bar{a} \\ 0 & \text{if } x < \underline{a} \text{ or } x > \bar{a} \end{cases}$$

and

$$v_{\tilde{a}}(x) = \begin{cases} [a - x + (x - \underline{a})u_{\tilde{a}}] / (a - \underline{a}) & \text{if } \underline{a} \leq x < a \\ u_{\tilde{a}} & \text{if } x = a \\ [x - a + (\bar{a} - x)u_{\tilde{a}}] / (\bar{a} - a) & \text{if } a < x \leq \bar{a} \\ 1 & \text{if } x < \underline{a} \text{ or } x > \bar{a} \end{cases}$$

respectively, where the values  $w_{\tilde{a}}$  and  $u_{\tilde{a}}$  represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy the following conditions:  $0 \leq w_{\tilde{a}} \leq 1$ ,  $0 \leq u_{\tilde{a}} \leq 1$  and  $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$ .

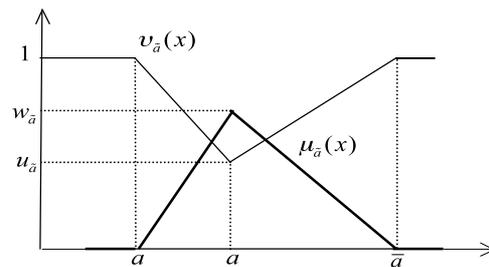


Fig. 1. An TIFN

Let  $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - v_{\tilde{a}}(x)$ , which is called as the intuitionistic fuzzy index of an element  $x$  in the TIFN  $\tilde{a}$ . It is the degree of indeterminacy membership of the element  $x$  to the TIFN  $\tilde{a}$ .

An TIFN  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} \rangle$  may express an ill-known quantify, which is approximately equal to  $a$ . Namely, the ill-known quantify is expressed using any value between  $\underline{a}$  and  $\bar{a}$  with different degree of membership and degree of non-membership. In other

words, the most possible value is  $a$  with the degree of membership  $w_{\bar{a}}$  and the degree of non-membership  $u_{\bar{a}}$ ; the pessimistic value is  $\underline{a}$  with the degree of membership 0 and the degree of non-membership 1; the optimistic value is  $\bar{a}$  with the degree of membership 0 and the degree of non-membership 1; other value is any  $x$  in the open interval  $(\underline{a}, \bar{a})$  with the degree of membership  $\mu_{\bar{a}}(x)$  and the degree of non-membership  $\nu_{\bar{a}}(x)$ .

An TIFN  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  mainly gives an approximately estimated range of an ill-known quantify, i.e., an interval  $[\underline{a}, \bar{a}]$ . For any fixed value  $x$  in the closed interval  $[\underline{a}, \bar{a}]$ , the degree of membership  $\mu_{\bar{a}}(x)$  and the degree of non-membership  $\nu_{\bar{a}}(x)$  are real numbers rather than intervals. However, for the interval-valued F-set  $B$ , the degree of membership  $\mu_B(x)$  of any fixed element  $x \in B$  is an interval rather than a real number.<sup>27,28</sup> Similarly, for any fixed element  $x \in B$ , the degree of non-membership is  $\nu_B(x) = 1 - \mu_B(x)$ , which is an interval rather than a real number. In other words, the TIFN  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  focuses on measurement uncertainty of the ill-known quantify estimated while the interval-valued F-set  $B$  focuses on uncertainty of the degree of belongingness to  $B$ . Therefore, the TIFN differs from the interval-valued F-set although the TIFN is a special IF-set on the real number set and the interval-valued F-set is mathematically equivalent to the IF-set.

Obviously, if  $\mu_{\bar{a}}(x) + \nu_{\bar{a}}(x) = 1$  then  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  is reduced to  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, 1 - w_{\bar{a}} \rangle$ , which is a triangular fuzzy number.<sup>7,29</sup>

It is easy to see that the definition of an TIFN is a generalization of that of the triangular fuzzy number.<sup>7,29</sup> Two new parameters  $w_{\bar{a}}$  and  $u_{\bar{a}}$  are introduced to reflect the confidence and non-confidence levels of the TIFN, respectively. Compared with the triangular fuzzy number, an TIFN may express more uncertainty.

**Definition 2.**<sup>30</sup> Let  $\tilde{a}$  and  $\tilde{b}$  be two TIFNs, denoted by  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$ ,  $\tilde{b} = \langle (\underline{b}, b, \bar{b}); w_{\bar{b}}, u_{\bar{b}} \rangle$  with  $w_{\bar{a}} \neq w_{\bar{b}}$  and  $u_{\bar{a}} \neq u_{\bar{b}}$ , the arithmetic operations are defined as follows:

- (i)  $\tilde{a} + \tilde{b} = \langle (\underline{a} + \underline{b}, a + b, \bar{a} + \bar{b}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle$ , where the symbols “ $\wedge$ ” and “ $\vee$ ” mean min and max operators, respectively,
- (ii)  $\tilde{a} - \tilde{b} = \langle (\underline{a} - \underline{b}, a - b, \bar{a} - \bar{b}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle$ ,

- (iii)  $\tilde{a} \times \tilde{b} = \begin{cases} \langle (\underline{ab}, ab, \bar{ab}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ \langle (\underline{ab}, ab, \bar{ab}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ \langle (\bar{ab}, ab, \underline{ab}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases}$
- (iv)  $\tilde{a} / \tilde{b} = \begin{cases} \langle (\underline{a}/\underline{b}, a/b, \bar{a}/\bar{b}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ \langle (\bar{a}/\bar{b}, a/b, \underline{a}/\underline{b}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ \langle (\bar{a}/\bar{b}, a/b, \underline{a}/\underline{b}); w_{\bar{a}} \wedge w_{\bar{b}}, u_{\bar{a}} \vee u_{\bar{b}} \rangle & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases}$
- (v)  $\lambda \tilde{a} = \begin{cases} \langle (\lambda \underline{a}, \lambda a, \lambda \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle & \text{if } \lambda > 0 \\ \langle (\lambda \bar{a}, \lambda a, \lambda \underline{a}); w_{\bar{a}}, u_{\bar{a}} \rangle & \text{if } \lambda < 0 \end{cases}$ , where  $\lambda$  is any real number,
- (vi)  $\tilde{a}^{-1} = \langle (1/\bar{a}, 1/a, 1/\underline{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$ .

**2.2. Cut sets and ranking order relation of TIFNs**

**Definition 3.** A  $(\alpha, \beta)$ -cut set of  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  is a crisp subset of  $\mathbb{R}$ , which is defined as  $\tilde{a}_{\alpha, \beta} = \{x \mid \mu_{\bar{a}}(x) \geq \alpha, \nu_{\bar{a}}(x) \leq \beta\}$ , where  $0 \leq \alpha \leq w_{\bar{a}}$ ,  $u_{\bar{a}} \leq \beta \leq 1$  and  $0 \leq \alpha + \beta \leq 1$ .

**Definition 4.** A  $\alpha$ -cut set of  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  is a crisp subset of  $\mathbb{R}$ , which is defined as follows:

$$\tilde{a}_{\alpha} = \{x \mid \mu_{\bar{a}}(x) \geq \alpha\}.$$

Using the membership function of  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  and Definition 4, it is easily seen that  $\tilde{a}_{\alpha} = \{x \mid \mu_{\bar{a}}(x) \geq \alpha\}$  is a closed interval and calculated as follows:

$$\tilde{a}_{\alpha} = [a + \alpha(a - \underline{a})/w_{\bar{a}}, a - \alpha(\bar{a} - a)/w_{\bar{a}}].$$

**Definition 5.** A  $\beta$ -cut set of  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  is a crisp subset of  $\mathbb{R}$ , which is defined as follows:

$$\tilde{a}_{\beta} = \{x \mid \nu_{\bar{a}}(x) \leq \beta\}.$$

Using the non-membership function of  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  and Definition 5, it is easily seen that  $\tilde{a}_{\beta} = \{x \mid \nu_{\bar{a}}(x) \leq \beta\}$  is a closed interval and calculated as follows:

$$\tilde{a}_{\beta} = [(1 - \beta)a + (\beta - u_{\bar{a}})\underline{a}]/(1 - u_{\bar{a}}), [(1 - \beta)a + (\beta - u_{\bar{a}})\bar{a}]/(1 - u_{\bar{a}})].$$

The following conclusion is easily reached from Definitions 3-5.

**Theorem 1.** Let  $\tilde{a} = \langle (\underline{a}, a, \bar{a}); w_{\bar{a}}, u_{\bar{a}} \rangle$  be any TIFN. For any  $\alpha \in [0, w_{\bar{a}}]$  and  $\beta \in [u_{\bar{a}}, 1]$ , where  $0 \leq \alpha + \beta \leq 1$ , the following equality is valid

$$\tilde{a}_{\alpha, \beta} = \tilde{a}_{\alpha} \cap \tilde{a}_{\beta}.$$

**Definition 6.** Let  $m(\tilde{a}_\alpha)$  and  $m(\tilde{a}_\beta)$  be mean values of the intervals  $\tilde{a}_\alpha$  and  $\tilde{a}_\beta$ , respectively, i.e.,

$$m(\tilde{a}_\alpha) = [2\alpha a + (w_{\tilde{a}} - \alpha)(\underline{a} + \bar{a})] / 2w_{\tilde{a}}$$

and

$$m(\tilde{a}_\beta) = [2(1 - \beta)a + (\beta - u_{\tilde{a}})(\underline{a} + \bar{a})] / 2(1 - u_{\tilde{a}}).$$

Then the average index of the membership function  $\mu_{\tilde{a}}(x)$  and the average index of the non-membership function  $\nu_{\tilde{a}}(x)$  for the TIFN  $\tilde{a}$  are defined as follows:

$$s_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} m(\tilde{a}_\alpha) d\alpha = w_{\tilde{a}}(2a + \underline{a} + \bar{a}) / 4$$

and

$$s_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 m(\tilde{a}_\beta) d\beta = (1 - u_{\tilde{a}})(2a + \underline{a} + \bar{a}) / 4,$$

respectively.

The ranking order relation of TIFNs is a difficult problem. In this paper, a new ranking order relation of TIFNs is defined based on Definition 6 as follows.

**Definition 7.** Assume that  $\tilde{a}$  and  $\tilde{b}$  be two TIFNs.  $s_\mu(\tilde{a}) = w_{\tilde{a}}(2a + \underline{a} + \bar{a}) / 4$  and  $s_\mu(\tilde{b}) = w_{\tilde{b}}(2b + \underline{b} + \bar{b}) / 4$  are the membership function average indexes of  $\tilde{a}$  and  $\tilde{b}$ , while  $s_\nu(\tilde{a}) = (1 - u_{\tilde{a}})(2a + \underline{a} + \bar{a}) / 4$  and  $s_\nu(\tilde{b}) = (1 - u_{\tilde{b}})(2b + \underline{b} + \bar{b}) / 4$  are the non-membership function average indexes of  $\tilde{a}$  and  $\tilde{b}$ , respectively. Then

- (i) If  $s_\mu(\tilde{a}) < s_\mu(\tilde{b})$  then  $\tilde{a}$  is smaller than  $\tilde{b}$ , denoted by  $\tilde{a} <_{IF} \tilde{b}$ ;
- (ii) If  $s_\mu(\tilde{a}) = s_\mu(\tilde{b})$  then
  - (a) If  $s_\nu(\tilde{a}) = s_\nu(\tilde{b})$  then  $\tilde{a}$  and  $\tilde{b}$  represent the same amount, i.e.,  $\tilde{a}$  is equal to  $\tilde{b}$ , denoted by  $\tilde{a} =_{IF} \tilde{b}$ ;
  - (b) If  $s_\nu(\tilde{a}) < s_\nu(\tilde{b})$  then  $\tilde{a}$  is smaller than  $\tilde{b}$ , denoted by  $\tilde{a} <_{IF} \tilde{b}$ .

The symbol " $<_{IF}$ " is an intuitionistic fuzzy version of the order relation " $<$ " in the real number set and has the linguistic interpretation "essentially less than". The symbols " $>_{IF}$ " and " $=_{IF}$ " are explained similarly.

### 3. Auxiliary programming models and lexicographic methodology for matrix games with payoffs of TIFNs

Let's consider the matrix games with payoffs of TIFNs. Assume that  $S_1 = \{\delta_1, \delta_2, \dots, \delta_m\}$  and  $S_2 = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$  are sets of pure strategies for two Players I and II,

respectively. The vectors  $\mathbf{y} = (y_1, y_2, \dots, y_m)^T$  and  $\mathbf{z} = (z_1, z_2, \dots, z_n)^T$  are mixed strategies for Players I and II, respectively, where  $y_i$  ( $i = 1, 2, \dots, m$ ) and  $z_j$  ( $j = 1, 2, \dots, n$ ) are probabilities in which Players I and II choose their pure strategies  $\delta_i \in S_1$  ( $i = 1, 2, \dots, m$ ) and  $\sigma_j \in S_2$  ( $j = 1, 2, \dots, n$ ), respectively. Sets of all mixed strategies for Players I and II are denoted by  $Y$  and  $Z$ , respectively, i.e.,

$$Y = \{\mathbf{y} = (y_1, y_2, \dots, y_m)^T \mid \sum_{i=1}^m y_i = 1, y_i \geq 0 (i = 1, 2, \dots, m)\}$$

and

$$Z = \{\mathbf{z} = (z_1, z_2, \dots, z_n)^T \mid \sum_{j=1}^n z_j = 1, z_j \geq 0 (j = 1, 2, \dots, n)\}.$$

Without loss of generality, the payoff matrix for the Player I is given by  $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ , where each  $\tilde{a}_{ij} = \langle (\underline{a}_{ij}, a_{ij}, \bar{a}_{ij}); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle$  ( $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ ) is an TIFN defined as above. Then a matrix game with payoffs of TIFNs is defined as a triplet IFG =  $\{Y, Z, \tilde{A}\}$ . In the following, we shall often call a matrix game with payoffs of TIFNs simply as an intuitionistic fuzzy matrix game IFG or IFG =  $\{Y, Z, \tilde{A}\}$ .

According to Definition 2, the intuitionistic fuzzy expectation payoff for Player I can be computed as follows:

$$\begin{aligned} \tilde{E}(\tilde{A}) &= \mathbf{y}^T \tilde{A} \mathbf{z} = \sum_{i=1}^m \sum_{j=1}^n \tilde{a}_{ij} y_i z_j \\ &= \langle (\sum_{i=1}^m \sum_{j=1}^n \underline{a}_{ij} y_i z_j, \sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i z_j, \sum_{i=1}^m \sum_{j=1}^n \bar{a}_{ij} y_i z_j); \wedge w_{\tilde{a}_{ij}}, \vee u_{\tilde{a}_{ij}} \rangle \end{aligned}$$

which is an TIFN.

As the intuitionistic fuzzy matrix game IFG is zero-sum, according to Definition 2, the intuitionistic fuzzy expectation payoff for Player II is obtained as follows:

$$\begin{aligned} \tilde{E}(-\tilde{A}) &= \mathbf{y}^T (-\tilde{A}) \mathbf{z} = \sum_{i=1}^m \sum_{j=1}^n (-\tilde{a}_{ij}) y_i z_j \\ &= \langle (-\sum_{i=1}^m \sum_{j=1}^n \bar{a}_{ij} y_i z_j, -\sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i z_j, -\sum_{i=1}^m \sum_{j=1}^n \underline{a}_{ij} y_i z_j); \wedge w_{\tilde{a}_{ij}}, \vee u_{\tilde{a}_{ij}} \rangle \end{aligned}$$

which is still an TIFN. Thus, in general, Player I's gain-floor and Player II's loss-ceiling should be TIFNs, denoted by  $\tilde{\mathbf{v}} = \langle (\underline{\mathbf{v}}, \mathbf{v}, \bar{\mathbf{v}}); w_{\tilde{\mathbf{v}}}, u_{\tilde{\mathbf{v}}} \rangle$  and  $\tilde{\omega} = \langle (\underline{\omega}, \omega, \bar{\omega}); w_{\tilde{\omega}}, u_{\tilde{\omega}} \rangle$ , respectively. Now, the concept of solutions of the intuitionistic fuzzy matrix game IFG is defined as follows.

**Definition 8.** (Reasonable solution of an intuitionistic fuzzy matrix game) Let  $\tilde{\mathbf{v}} = \langle (\underline{\mathbf{v}}, \mathbf{v}, \bar{\mathbf{v}}); w_{\tilde{\mathbf{v}}}, u_{\tilde{\mathbf{v}}} \rangle$  and  $\tilde{\omega} = \langle (\underline{\omega}, \omega, \bar{\omega}); w_{\tilde{\omega}}, u_{\tilde{\omega}} \rangle$  be TIFNs. Assume that there

exist  $\mathbf{y}^* \in Y$  and  $\mathbf{z}^* \in Z$ . Then  $(\mathbf{y}^*, \mathbf{z}^*, \tilde{v}, \tilde{w})$  is called a reasonable solution of the intuitionistic fuzzy matrix game IFG if for any  $\mathbf{z} \in Z$  and  $\mathbf{y} \in Y$ ,  $\mathbf{y}^*$  and  $\mathbf{z}^*$  satisfy both  $\mathbf{y}^{*T} \tilde{A} \mathbf{z} \geq_{IF} \tilde{v}$  and  $\mathbf{y}^T \tilde{A} \mathbf{z}^* \leq_{IF} \tilde{w}$ . If  $(\mathbf{y}^*, \mathbf{z}^*, \tilde{v}, \tilde{w})$  is a reasonable solution of the IFG then  $\tilde{v}$  and  $\tilde{w}$  are called reasonable values of Players I and II,  $\mathbf{y}^*$  and  $\mathbf{z}^*$  are called reasonable strategies for Players I and II, respectively.

It is worth noticing that Definition 8 only gives the notion of a reasonable solution rather than an optimal solution. Let  $V$  and  $W$  be the sets of all reasonable game values  $\tilde{v}$  and  $\tilde{w}$  for Players I and II, respectively.

**Definition 9.** (Solution of an intuitionistic fuzzy matrix game) Assume that there exist  $\tilde{v}^* \in V$  and  $\tilde{w}^* \in W$ . If there do not exist any  $\tilde{v} \in V$  ( $\tilde{v} \neq \tilde{v}^*$ ) and  $\tilde{w} \in W$  ( $\tilde{w} \neq \tilde{w}^*$ ) such that  $\tilde{v}^* \leq_{IF} \tilde{v}$  and  $\tilde{w}^* \geq_{IF} \tilde{w}$  are satisfied, then  $(\mathbf{y}^*, \mathbf{z}^*, \tilde{v}^*, \tilde{w}^*)$  is called a solution of the intuitionistic fuzzy matrix game IFG, where  $\mathbf{y}^*$  and  $\mathbf{z}^*$  are called the maximin strategy and minimax strategy for Players I and II, respectively;  $\tilde{v}^*$  and  $\tilde{w}^*$  are called Player I's gain-floor and Player II's loss-ceiling, respectively.  $\mathbf{y}^{*T} \tilde{A} \mathbf{z}^*$  is called the value of the IFG.

According to Definitions 8 and 9, the maximin strategy  $\mathbf{y}^* \in Y$  for Player I and minimax strategy  $\mathbf{z}^* \in Z$  for Player II can be generated by solving a pair of intuitionistic fuzzy mathematical programming models constructed as follows:

$$\begin{aligned} & \max \{\tilde{v}\} \\ & \left. \begin{aligned} & \sum_{i=1}^m \tilde{a}_{ij} y_i z_j \geq_{IF} \tilde{v} \text{ for any } \mathbf{z} \in Z \\ & \sum_{i=1}^m y_i = 1 \\ & y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{aligned} \right\} \quad (1) \end{aligned}$$

and

$$\begin{aligned} & \min \{\tilde{w}\} \\ & \left. \begin{aligned} & \sum_{j=1}^n \tilde{a}_{ij} y_i z_j \leq_{IF} \tilde{w} \text{ for any } \mathbf{y} \in Y \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right\} \quad (2) \end{aligned}$$

respectively, where  $\tilde{v}$  and  $\tilde{w}$  are TIFN variables.

It makes sense to consider only the extreme points of sets  $Y$  and  $Z$  in the constraints of Eqs. (1) and (2) since " $\leq_{IF}$ " and " $\geq_{IF}$ " preserve the ranking order relations when TIFNs are multiplied by positive numbers according to Definition 2. Therefore, Eqs. (1) and (2) can be converted into the intuitionistic fuzzy mathematical programming models as follows:

$$\begin{aligned} & \max \{\tilde{v}\} \\ & \left. \begin{aligned} & \sum_{i=1}^m \tilde{a}_{ij} y_i \geq_{IF} \tilde{v} \quad (j = 1, 2, \dots, n) \\ & \sum_{i=1}^m y_i = 1 \\ & y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{aligned} \right\} \quad (3) \end{aligned}$$

and

$$\begin{aligned} & \min \{\tilde{w}\} \\ & \left. \begin{aligned} & \sum_{j=1}^n \tilde{a}_{ij} z_j \leq_{IF} \tilde{w} \quad (i = 1, 2, \dots, m) \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right\} \quad (4) \end{aligned}$$

respectively.

In this study, the intuitionistic fuzzy optimization is made in the sense of Definitions 6 and 7. In the following, we will focus on studying the solution method and procedure of Eqs. (3) and (4).

Using Definition 7, Eq. (3) can be transformed into the bi-objective programming problem as follows:

$$\begin{aligned} & \max \{s_\mu(\tilde{v}), s_\nu(\tilde{v})\} \\ & \left. \begin{aligned} & s_\mu \left( \sum_{i=1}^m \tilde{a}_{ij} y_i \right) \geq s_\mu(\tilde{v}) \quad (j = 1, 2, \dots, n) \\ & s_\nu \left( \sum_{i=1}^m \tilde{a}_{ij} y_i \right) \geq s_\nu(\tilde{v}) \quad (j = 1, 2, \dots, n) \\ & \sum_{i=1}^m y_i = 1 \\ & y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{aligned} \right\} \quad (5) \end{aligned}$$

According to Definition 6, Eq. (5) can be rewritten as follows:

$$\begin{aligned} & \max \{w_{\underline{v}}(\underline{v} + 2\nu + \bar{v})/4, (1-u_{\bar{v}})(\underline{v} + 2\nu + \bar{v})/4\} \\ & \left\{ \begin{array}{l} \sum_{i=1}^m w_{\bar{a}_{ij}}(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq w_{\underline{v}}(\underline{v} + 2\nu + \bar{v})/4 \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^m (1-u_{\bar{a}_{ij}})(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq (1-u_{\bar{v}})(\underline{v} + 2\nu + \bar{v})/4 \quad (j = 1, 2, \dots, n) \end{array} \right. \\ & \text{s.t.} \left\{ \begin{array}{l} \underline{v} \leq \nu \\ \nu \leq \bar{v} \\ \sum_{i=1}^m y_i = 1 \\ y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{array} \right. \end{aligned} \tag{6}$$

where  $\underline{v}$ ,  $\nu$ ,  $\bar{v}$  and  $y_i$  ( $i = 1, 2, \dots, m$ ) are variables, and  $w_{\underline{v}} = \wedge w_{\bar{a}_{ij}}$  and  $u_{\bar{v}} = \vee u_{\bar{a}_{ij}}$ .

In general, it may be difficult to obtain the exact  $\bar{v}^*$ . The reason is that determining  $\bar{v}^*$  exactly needs to compute all three variables  $\underline{v}$ ,  $\nu$ ,  $\bar{v}$ . Therefore, in order to compute Eq. (6) easily, let  $\nu_1 = w_{\underline{v}}(\underline{v} + 2\nu + \bar{v})/4$  and  $\nu_2 = (1-u_{\bar{v}})(\underline{v} + 2\nu + \bar{v})/4$ . Then, Eq. (6) can be rewritten as follows:

$$\begin{aligned} & \max \{v_1, v_2\} \\ & \left\{ \begin{array}{l} \sum_{i=1}^m w_{\bar{a}_{ij}}(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq v_1 \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^m (1-u_{\bar{a}_{ij}})(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq v_2 \quad (j = 1, 2, \dots, n) \end{array} \right. \\ & \text{s.t.} \left\{ \begin{array}{l} v_2 \geq v_1 \\ \sum_{i=1}^m y_i = 1 \\ y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{array} \right. \end{aligned} \tag{7}$$

Obviously, Eq. (7) is a bi-objective linear programming model on the decision variables  $v_1$ ,  $v_2$  and  $y_i$  ( $i = 1, 2, \dots, m$ ). There are few standard ways of defining a solution of multiobjective programming. Normally, the concept of Pareto optimal/efficient solutions is commonly-used. There exist several solution methods for them.<sup>31,32</sup> However, in this study we focus on developing a lexicographic approach to solve Eq. (7) in the sense of Pareto optimality.

**Level 1** Usually the average index of the membership function is more important than that of the non-membership function. Thus, according to Eq. (7), a linear programming model at the first level can be easily constructed as follows:

$$\begin{aligned} & \max \{v_1\} \\ & \left\{ \begin{array}{l} \sum_{i=1}^m w_{\bar{a}_{ij}}(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq v_1 \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^m (1-u_{\bar{a}_{ij}})(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq v_2 \quad (j = 1, 2, \dots, n) \end{array} \right. \\ & \text{s.t.} \left\{ \begin{array}{l} v_2 \geq v_1 \\ \sum_{i=1}^m y_i = 1 \\ y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{array} \right. \end{aligned} \tag{8}$$

where  $\mathbf{y} = (y_1, y_2, \dots, y_m)^T$ ,  $v_1$  and  $v_2$  are decision variables. Using the existing Simplex method for linear programming, an optimal solution of Eq. (8) is obtained, denoted by  $(\mathbf{y}^0, v_1^0, v_2^0)$ .

**Level 2** According to the lexicographic method, combining with Eq. (7) and the solution of Eq. (8), a linear programming model at the second level can be constructed as follows:

$$\begin{aligned} & \max \{v_2\} \\ & \left\{ \begin{array}{l} \sum_{i=1}^m w_{\bar{a}_{ij}}(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq v_1 \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^m (1-u_{\bar{a}_{ij}})(\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij})y_i / 4 \geq v_2 \quad (j = 1, 2, \dots, n) \end{array} \right. \\ & \text{s.t.} \left\{ \begin{array}{l} v_2 \geq v_1 \\ v_1 \geq v_1^0 \\ v_2 \geq v_2^0 \\ \sum_{i=1}^m y_i = 1 \\ y_i \geq 0 \quad (i = 1, 2, \dots, m) \end{array} \right. \end{aligned} \tag{9}$$

where  $\mathbf{y} = (y_1, y_2, \dots, y_m)^T$ ,  $v_1$  and  $v_2$  are decision variables.

In Eq. (9), adding the constraints  $v_1 \geq v_1^0$  and  $v_2 \geq v_2^0$  aims to improve  $v_1$  and  $v_2$ . This is the reason why the second level linear programming (i.e., Eq. (9)) is introduced after the first level linear programming (i.e., Eq. (8)).

Using the existing Simplex method for linear programming, an optimal solution of Eq. (9) is obtained, denoted by  $(\mathbf{y}^*, v_1^*, v_2^*)$ .

It is not difficult to prove that  $(\mathbf{y}^*, v_1^*, v_2^*)$  is a Pareto optimal solution of Eq. (7). Thus, the maximin strategy  $\mathbf{y}^*$  for Player I is obtained as well as the membership

degree average index  $v_1^*$  and the non-membership degree average index  $v_2^*$  of Player I's gain-floor  $\tilde{v}^*$ .

Similarly, according to Definition 7, Eq. (4) can be transformed into the bi-objective programming problem as follows:

$$\begin{aligned} & \min \{s_\mu(\tilde{\omega}), s_\nu(\tilde{\omega})\} \\ & s.t. \left\{ \begin{aligned} & s_\mu \left( \sum_{j=1}^n \tilde{a}_{ij} z_j \right) \leq s_\mu(\tilde{\omega}) \quad (i = 1, 2, \dots, m) \\ & s_\nu \left( \sum_{j=1}^n \tilde{a}_{ij} z_j \right) \leq s_\nu(\tilde{\omega}) \quad (i = 1, 2, \dots, m) \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right. \quad (10) \end{aligned}$$

According to Definition 6, Eq. (10) can be rewritten as follows:

$$\begin{aligned} & \min \{w_{\tilde{\omega}}(\underline{\omega} + 2\omega + \bar{\omega})/4, (1 - u_{\tilde{\omega}})(\underline{\omega} + 2\omega + \bar{\omega})/4\} \\ & s.t. \left\{ \begin{aligned} & \sum_{j=1}^n w_{\tilde{a}_{ij}} (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq w_{\tilde{\omega}}(\underline{\omega} + 2\omega + \bar{\omega})/4 \quad (i = 1, 2, \dots, m) \\ & \sum_{j=1}^n (1 - u_{\tilde{a}_{ij}}) (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq (1 - u_{\tilde{\omega}})(\underline{\omega} + 2\omega + \bar{\omega})/4 \quad (i = 1, 2, \dots, m) \\ & \underline{\omega} \leq \omega \\ & \omega \leq \bar{\omega} \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right. \quad (11) \end{aligned}$$

where  $\underline{\omega}$ ,  $\omega$ ,  $\bar{\omega}$  and  $z_j$  ( $j = 1, 2, \dots, n$ ) are variables, and  $w_{\tilde{\omega}} = \wedge w_{\tilde{a}_{ij}}$  and  $u_{\tilde{\omega}} = \vee u_{\tilde{a}_{ij}}$ .

In a similar consideration to Eq. (6), let  $\omega_1 = w_{\tilde{\omega}}(\underline{\omega} + 2\omega + \bar{\omega})/4$  and  $\omega_2 = (1 - u_{\tilde{\omega}})(\underline{\omega} + 2\omega + \bar{\omega})/4$ , Eq. (11) can be simply rewritten as follows:

$$\begin{aligned} & \min \{ \omega_1, \omega_2 \} \\ & s.t. \left\{ \begin{aligned} & \sum_{j=1}^n w_{\tilde{a}_{ij}} (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq \omega_1 \quad (i = 1, 2, \dots, m) \\ & \sum_{j=1}^n (1 - u_{\tilde{a}_{ij}}) (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq \omega_2 \quad (i = 1, 2, \dots, m) \\ & \omega_1 \leq \omega_2 \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right. \quad (12) \end{aligned}$$

Thus, solving Eq. (12) turns into solving Eqs. (13) and (14) successively according to the lexicographic method. Namely,

**Level 1** A linear programming model at the first level is constructed as follows:

$$\begin{aligned} & \min \{ \omega_1 \} \\ & s.t. \left\{ \begin{aligned} & \sum_{j=1}^n w_{\tilde{a}_{ij}} (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq \omega_1 \quad (i = 1, 2, \dots, m) \\ & \sum_{j=1}^n (1 - u_{\tilde{a}_{ij}}) (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq \omega_2 \quad (i = 1, 2, \dots, m) \\ & \omega_1 \leq \omega_2 \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right. \quad (13) \end{aligned}$$

where  $\mathbf{z} = (z_1, z_2, \dots, z_n)^T$ ,  $\omega_1$  and  $\omega_2$  are decision variables. Let an optimal solution of Eq. (13) be denoted by  $(\mathbf{z}^0, \omega_1^0, \omega_2^0)$ .

**Level 2** A linear programming model at the second level is constructed as follows:

$$\begin{aligned} & \min \{ \omega_2 \} \\ & s.t. \left\{ \begin{aligned} & \sum_{j=1}^n w_{\tilde{a}_{ij}} (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq \omega_1 \quad (i = 1, 2, \dots, m) \\ & \sum_{j=1}^n (1 - u_{\tilde{a}_{ij}}) (\underline{a}_{ij} + 2a_{ij} + \bar{a}_{ij}) z_j / 4 \leq \omega_2 \quad (i = 1, 2, \dots, m) \\ & \omega_1 \leq \omega_2 \\ & \omega_1 \leq \omega_1^0 \\ & \omega_2 \leq \omega_2^0 \\ & \sum_{j=1}^n z_j = 1 \\ & z_j \geq 0 \quad (j = 1, 2, \dots, n) \end{aligned} \right. \quad (14) \end{aligned}$$

Let an optimal solution of Eq. (14) be denoted by  $(\mathbf{z}^*, \omega_1^*, \omega_2^*)$ . It is not difficult to prove that  $(\mathbf{z}^*, \omega_1^*, \omega_2^*)$  is a Pareto optimal solution of Eq. (12). Thus, the minimax strategy  $\mathbf{z}^*$  can be obtained as well as the average index  $\omega_1^*$  of the membership degree and the average index  $\omega_2^*$  of the non-membership degree for Player II's loss-ceiling  $\omega^*$ .

It is easy to see that  $v_1 = v_2$  and  $\omega_1 = \omega_2$  if  $w_{\tilde{a}_{ij}} = 1$  and  $u_{\tilde{a}_{ij}} = 0$ , i.e., all  $\tilde{a}_{ij} = \langle (\underline{a}_{ij}, a_{ij}, \bar{a}_{ij}); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle$  ( $i = 1, 2, \dots, m$ ;  $j = 1, 2, \dots, n$ ) are triangular fuzzy numbers. Thus, the linear programming models (i.e., Eqs. (7) and (12)) of matrix games with payoffs of TIFNs are degenerated to those of matrix games with payoffs of triangular fuzzy numbers.<sup>1,3</sup> Therefore, the

intuitionistic fuzzy matrix game is a natural generalization of the matrix game with payoffs of triangular fuzzy numbers.

#### 4. An application to the market share problem

Suppose that there are two companies  $p_1$  and  $p_2$  aiming to enhance the market share of a product in a targeted market under the circumstance that the demand amount of the product in the targeted market basically is fixed. In other words, the market share of one company increases while the market share of another company decreases. The two companies are considering about two strategies to increase the market share: strategy  $\delta_1$  (advertisement),  $\delta_2$  (reduce the price). The above problem may be regarded as a matrix game. Namely, the companies  $p_1$  and  $p_2$  are regarded as Players I and II, respectively. They may use strategies  $\delta_1$  and  $\delta_2$ . Due to a lack of information or imprecision of the available information, the managers of the two companies usually are not able to exactly forecast the sales amount of the companies. They estimate the sales amount with a certain confidence degree, but it is possible that they are not so sure about it. Thus, there may be a hesitation about the estimation of the sales amount. In order to handle the uncertain situation, TIFNs are used to express the sales amount of the product. The payoff matrix  $\tilde{A}$  for the company  $p_1$  is given as follows:

$$\tilde{A} = \begin{pmatrix} \langle(175,180,190);0.6,0.2\rangle & \langle(150,156,158);0.6,0.1\rangle \\ \langle(80,90,100);0.9,0.1\rangle & \langle(175,180,190);0.6,0.2\rangle \end{pmatrix}$$

where  $\langle(175,180,190);0.6,0.2\rangle$  in the matrix  $\tilde{A}$  is an TIFN, which indicates that the sales amount of the company  $p_1$  is about 180 when the companies  $p_1$  and  $p_2$  use the strategy  $\delta_1$  (advertisement) simultaneously. The maximum confidence degree of the manager is 0.6 while the minimum non-confidence degree of the manager is 0.2. In other words, his hesitation degree is 0.2. Other elements (i.e., TIFNs) in the matrix  $\tilde{A}$  are explained similarly.

According to Eq. (8), the linear programming model is obtained as follows:

$$\begin{aligned} & \max \{v_1\} \\ & \begin{cases} 0.6(175 + 2 \times 180 + 190)y_1 / 4 + 0.9(80 + 2 \times 90 + 100)y_2 / 4 \geq v_1 \\ 0.6(150 + 2 \times 156 + 158)y_1 / 4 + 0.6(175 + 2 \times 180 + 190)y_2 / 4 \geq v_1 \\ 0.8(175 + 2 \times 180 + 190)y_1 / 4 + 0.9(80 + 2 \times 90 + 100)y_2 / 4 \geq v_2 \\ 0.9(150 + 2 \times 156 + 158)y_1 / 4 + 0.8(175 + 2 \times 180 + 190)y_2 / 4 \geq v_2 \end{cases} \\ & \text{s.t.} \begin{cases} v_2 \geq v_1 \\ y_1 + y_2 = 1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases} \end{aligned} \tag{15}$$

Solving Eq. (15) using the Simplex method for linear programming, an optimal solution  $(y^0, v_1^0, v_2^0)$  can be obtained, where  $y^0 = (0.638, 0.362)^T$ ,  $v_1^0 = 98.70$  and  $v_2^0 = 98.70$ .

According to Eq. (9), the linear programming problem can be obtained as follows:

$$\begin{aligned} & \max \{v_2\} \\ & \begin{cases} 0.6(175 + 2 \times 180 + 190)y_1 / 4 + 0.9(80 + 2 \times 90 + 100)y_2 / 4 \geq v_1 \\ 0.6(150 + 2 \times 156 + 158)y_1 / 4 + 0.6(175 + 2 \times 180 + 190)y_2 / 4 \geq v_1 \\ 0.8(175 + 2 \times 180 + 190)y_1 / 4 + 0.9(80 + 2 \times 90 + 100)y_2 / 4 \geq v_2 \\ 0.9(150 + 2 \times 156 + 158)y_1 / 4 + 0.8(175 + 2 \times 180 + 190)y_2 / 4 \geq v_2 \end{cases} \\ & \text{s.t.} \begin{cases} v_2 \geq v_1 \\ v_1 \geq 98.70 \\ v_2 \geq 98.70 \\ y_1 + y_2 = 1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases} \end{aligned} \tag{16}$$

Solving Eq. (16) using the Simplex method for linear programming, an optimal solution  $(y^*, v_1^*, v_2^*)$  can be obtained, where  $y^* = (0.638, 0.362)^T$ ,  $v_1^* = 98.70$  and  $v_2^* = 121.84$ . Therefore, the maximin strategy for Player I and the membership degree and non-membership degree average indexes for Player I's gain-floor are  $y^*$ ,  $v_1^*$  and  $v_2^*$ , respectively.

Similarly, according to Eqs. (13) and (14), the linear programming models can be obtained as follows:

$$\begin{aligned} & \min \{\omega_1\} \\ & \begin{cases} 0.6(175 + 2 \times 180 + 190)z_1 / 4 + 0.6(150 + 2 \times 156 + 158)z_2 / 4 \leq \omega_1 \\ 0.9(80 + 2 \times 90 + 100)z_1 / 4 + 0.6(175 + 2 \times 180 + 190)z_2 / 4 \leq \omega_1 \\ 0.8(175 + 2 \times 180 + 190)z_1 / 4 + 0.9(150 + 2 \times 156 + 158)z_2 / 4 \leq \omega_2 \\ 0.9(80 + 2 \times 90 + 100)z_1 / 4 + 0.8(175 + 2 \times 180 + 190)z_2 / 4 \leq \omega_2 \end{cases} \\ & \text{s.t.} \begin{cases} \omega_1 \leq \omega_2 \\ z_1 + z_2 = 1 \\ z_1 \geq 0, z_2 \geq 0 \end{cases} \end{aligned} \tag{17}$$

and

$$\begin{aligned} & \min\{\omega_2\} \\ & \begin{cases} 0.6(175 + 2 \times 180 + 190)z_1 / 4 + 0.6(150 + 2 \times 156 + 158)z_2 / 4 \leq \omega_1 \\ 0.9(80 + 2 \times 90 + 100)z_1 / 4 + 0.6(175 + 2 \times 180 + 190)z_2 / 4 \leq \omega_1 \\ 0.8(175 + 2 \times 180 + 190)z_1 / 4 + 0.9(150 + 2 \times 156 + 158)z_2 / 4 \leq \omega_2 \\ 0.9(80 + 2 \times 90 + 100)z_1 / 4 + 0.8(175 + 2 \times 180 + 190)z_2 / 4 \leq \omega_2 \end{cases} \\ & \text{s.t.} \begin{cases} \omega_1 \leq 98.70 \\ \omega_2 \leq 141.49 \\ z_1 + z_2 = 1 \\ z_1 \geq 0, z_2 \geq 0 \end{cases} \end{aligned} \tag{18}$$

respectively. Solving Eqs. (17) and (18), the minimax strategy for Player II and the membership degree and non-membership degree average indexes for II’s loss-ceiling can be obtained as  $z^* = (0.362, 0.638)^T$ ,  $\omega_1^* = 98.703$  and  $\omega_2^* = 141.49$ , respectively.

It is easy to see that the value of the intuitionistic fuzzy matrix game IFG is

$$E(y^*, z^*) = y^{*T} A z^* = \langle (152.37, 158.44, 165.18); 0.6, 0.2 \rangle$$

depicted as in Fig. 2, which is an TIFN and indicates that the sales amount of the company  $p_1$  is approximately equal to 158.44 when the companies  $p_1$  and  $p_2$  choose the mixed strategies  $y^* = (0.638, 0.362)^T$  and  $z^* = (0.362, 0.638)^T$ , respectively. For the manager of the company  $p_1$ , the maximum confidence degree about the sales amount “approximately 158.44” is 0.6 while the minimum non-confidence degree about the sales amount “approximately 158.44” is 0.2, i.e., his hesitation degree about the sales amount “approximately 158.44” is 0.2. The obtained results show that the IF-set may express information more abundant and flexible than the FS when it is used to deal with uncertainty in game problems.

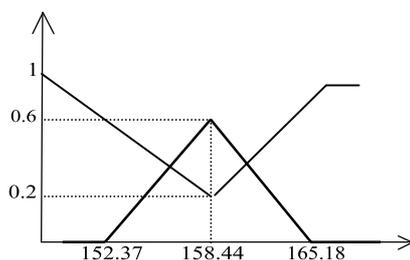


Fig. 2. The solution of the intuitionistic fuzzy matrix game for the market share problem

### 5. Conclusion

We have developed the concept of TIFNs, operations, cut sets and the ranking order relations as well as the

concept of solutions of matrix games with payoffs of TIFNs. A pair of intuitionistic fuzzy optimization models are established for two Players, which are transformed into bi-objective linear programming models based on the ranking order relations of TIFNs. Using the lexicographic method, two simpler auxiliary linear programming models are constructed to generate the maximin and minimax strategies for Players and the value of the intuitionistic fuzzy matrix game. It has been shown that the models of the intuitionistic fuzzy matrix game proposed in this paper extend those of the fuzzy matrix game.

Although the proposed method is illustrated with the market share problem, it may also be applied to similar competitive decision/game problems using IF-sets with a wide spectrum of possibilities, which enables the satisfying strategies that Players can expect.

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