Abstract—Fault detection and diagnosis (FDD) for singular stochastic distribution control (SDC) systems via the output probability density functions (PDFs) have been discussed. The PDFs can be approximated via square-root B-spline expansion, and expansions to represent the dynamics weighting systems between the system input and output are established to formulate the FTC problem. In Section 3, the PDFs can be approximated via square-root B-spline expansion, and expansions to represent the dynamics weighting systems between the system input and output.

I. INTRODUCTION

Fault detection and diagnosis (FDD) are important research areas for improving control systems reliability. Many effective methods have been presented in the past two decades for stochastic systems. Up to now, most of the existing FDD algorithms have only been concerned with Gaussian distribution systems. However, nonlinearity may lead to non-Gaussian output, where mean and variance of the system output are insufficient to characterize their statistical behavior precisely ([11], [12], [13]). As such, there are new FDD algorithms that are needed to develop, which can be applied to the stochastic systems subjected to non-Gaussian SDC systems. This forms the main purpose of the current work.

This paper is organized as follows. In Section 2, the output PDFs expansion and the nonlinear weight dynamic are established to formulate the FTC problem. In Section 3, the FTC filtering is designed to compensate or reject fault. The concluding remarks are presented in Section 4.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a continuous time dynamic stochastic distribution systems where $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in [a,b]$ represents the system output, and $F$ is the fault to be compensated or rejected, a typical example of which is an actuator fault. At any time $t$, the probability of output $y(t)$ lying inside $[a,b]$ can be described as

$$\int_{a}^{b} \phi(z,u(t),F) dz$$

where $\phi(z,u(t),F)(i=1,2,\ldots,n)$ are the corresponding weights. $b(z)(i=1,2,\ldots,n)$ are a pre-specified basis function, and $\omega_0(z,u(t),F)$ stands for the model uncertainty or the error, which is supposed to satisfy $|\omega_0(z,u(t),F)| \leq \delta_0$, $\delta_0$ is a known positive constant.

Denote

$$B_0(z) = [b_1(z), b_2(z), \ldots, b_n(z)]$$

and

$$A_0 = \sum_{i=1}^{n} b_i^T(z)b_i$$

For the following model is given:

$$h(V(t)) = B(z)\tilde{V}(t)g(z,u(t),F)$$

where

$$h(V(t)) = \frac{1}{A_\lambda}[-A_\lambda V(t) + \sqrt{A_\lambda^2 t - V^T(t)A_\lambda V(t)}]$$

From the boundedness of $\omega_0(z,u(t),F)$, it can be assumed...
that |ω(z,u(t),F)| ≤ δ holds for all \{z,u(t),F\}, where δ is a known positive constant.

In this paper the nonlinear dynamic model will be considered as follows

\[
\begin{bmatrix}
E(t) = Ax(t) + Gg(x(t)) + Du(t) + F \\
V(t) = Cx(t)
\end{bmatrix}
\] (4)

where x(t) ∈ \mathbb{R}^m is the unmeasured state, and A, G, D, C represent the known parametric matrices of the dynamic part of the weight system. In fact, these matrices can be obtained either by physical modeling or the scaling estimation technique described in [1] and [6]. E ∈ \mathbb{R}^{m \times m} is a known singular matrix, i.e., rank(E) = r < m; g(x(t)) ∈ \mathbb{R}^r is a nonlinear vector function that represents the nonlinear dynamics of the weight model and is supposed to satisfy g(0) = 0.

The following network can be used to approximate the continuous unknown function F(t) := F(x,u)

\[
F(x,u) = TWS(x,u) + θ(x,u)
\] (5)

where T is given matrix, W is the ideal weight matrix, θ(x,u) is a neural network approximation error, S(x,u) is the basis function. Since the state x is immeasurable, the output of neural network can be expressed as

\[
\hat{F}(\hat{x},u) = TWS(\hat{x},u)
\] (6)

where \(\hat{x}(t)\) is the estimated state, \(\hat{W}_i\) is an estimated matrix.

### III. FAULT DETECTION AND DIAGNOSIS VIA OUTPUT PDFS

#### A. Observer-based fault detection

Since the measured information is the output probability distribution, in order to detect the fault based on the changes of output PDFs, the following full-order observer is applied to detect the fault.

\[
\begin{bmatrix}
\dot{E}(t) = A\hat{x}(t) + Gg(\hat{x}(t)) + Du(t) + Ls(t) \\
\dot{e}(t) = \int_0^t \sigma(z)\left[\sqrt{p(z,u(t),F)} - \sqrt{p(z,u(t),F)}\right]dz
\end{bmatrix}
\] (7)

where \(\hat{x}(t)\) is the estimated state, \(L ∈ \mathbb{R}^{m \times m}\) is the gain to be determined and the residual \(e(t)\) is formulated as an integral of the difference between the measured PDFs and the estimated ones, \(σ(z) ∈ \mathbb{R}^{m}\) can be regarded as a pre-specified weight vector lying [a,b] and makes the integration simple or adjust the scale of \(σ(z)\).

By defining

\[
e(t) = x(t) - \hat{x}(t)\]

\[
\hat{g}(t) = g(x(t)) - g(\hat{x}(t))\]

\(\hat{h}(t) = h(Cx(t)) - h(C\hat{x}(t))\), the estimation error system can be described

\[
\begin{bmatrix}
\dot{E}(t) = (A - LI_1)e(t) + G\hat{g}(t) - LI_2\hat{h}(t) - L\Delta(t) + F \\
\dot{e}(t) = \int_0^t \sigma(z)\left[\sqrt{p(z,u(t),F)} - \sqrt{p(z,u(t),F)}\right]dz
\end{bmatrix}
\] (8)

where

\[
\Gamma_1 = \int_0^t \sigma(z)B_1(z)Edt, \Gamma_2 = \int_0^t \sigma(z)B_2(z)Edt, \Gamma_3 = \int_0^t \sigma(z)B_3(z)Edt\]

It can be seen that

\[
e(t) = \Gamma_1 e(t) + \Gamma_2 \hat{h}(t) + \Delta(t)
\] (9)

From

\[
\|\Delta(t)\| = \int_0^t \sigma(z)\left\|\alpha(z, u(t), F)d\right\| \leq \alpha \quad \forall \alpha = \delta \int_0^t \sigma(z)dz
\]

E is a singular matrix, hence exist two orthogonal matrices U and V such that

\[
UEV = \begin{bmatrix}
\Sigma & 0 \\
0 & 0
\end{bmatrix}
\] (10)

where \(\Sigma = diag(λ_1, \ldots, λ_n)\) and λ\(_i\) > 0 are the singular values of singular matrix E.

As shown in reference [7] and [8], Denote

\[
U(A - LI_1)V = \begin{bmatrix}
A_1 & A_2 \\
A_2 & \epsilon(t)
\end{bmatrix} = V^{-1}(t)
\]

\[
UG = [G_1 G_2]^TUL_1 = [H_1 H_2]UL = [L_1 L_2]^T.
\] (11)

#### Assumption 1. If A\(_{22}\) is invertible, then the following inequality

\[
\|A_2\|_{K(A_{22})} > \gamma \|G_2\| + \delta \|C\|\|H_2\|
\]

holds, where K(A\(_{22}\)) is the condition number of A\(_{22}\).

In the absence of F, Eq.(4) is transformed into

\[
\begin{bmatrix}
\Delta e(t) = A_1e(t) + A_2\epsilon(t) + G_1\hat{g}(t) - H_1\hat{h}(t) - L_1\Delta(t) \\
0 = A_1e(t) + A_2\epsilon(t) + G_2\hat{g}(t) - H_2\hat{h}(t) - L_2\Delta(t)
\end{bmatrix}
\] (12)

First, when there is no fault, our objective is to find L such that the system (12) is stable, which can be formulated in the following theorem.

#### Theorem 1. For the parameters γ > 0 and \(κ_\gamma > 0\) (i = 1, 2), if there exist matrices R and P with P being nonsingular, satisfying

\[
E^2P = P^2E \geq 0
\] (13)

\[
\begin{bmatrix}
\Pi & P^2G & R^T \gamma U_i^T \gamma U_i & U_i
\end{bmatrix}
\]

\[
\begin{bmatrix}
* & -\frac{1}{κ_1} & 0 & 0 & 0 \\
* & * & -\frac{1}{κ_2} & 0 & 0 \\
* & * & * & -κ_1 & 0 \\
* & * & * & * & -κ_2
\end{bmatrix}
\] (14)

where \(\Pi = A^T P + P^T A - R^T \gamma R - \Gamma_1 \gamma R^T + \gamma I\), then in the absence of F(x,u), the estimated system (11) with gain L = P\(^T\)R is asymptotically stable.

#### Proof. Substituting Eq.(10) and (11) into Eq.(13), it can be seen that \(Σ_i = P_1^T Σ_j > 0\) and \(P_{12} = 0\). A\(_{21}\) and A\(_{22}\) are invertible.

Define the Lyapunov candidate function as follows

\[
\begin{align*}
& V(0) = \epsilon(0)^T P \epsilon(0) + k_1 \epsilon(0)^T C_1 \hat{e}(0) + k_2 \epsilon(0)^T C_2 \hat{h}(0) + k_3 \epsilon(0)^T C_3 \hat{h}(0) \\
& = \epsilon(0)^T P \epsilon(0) + k_1 \epsilon(0)^T C_1 \hat{e}(0) + k_2 \epsilon(0)^T C_2 \hat{h}(0) + k_3 \epsilon(0)^T C_3 \hat{h}(0)
\end{align*}
\]

Along the trajectories of (12) in the absence of F, it can be shown that

\[
\dot{V}(t) = \epsilon(t)^T [(A - LI_1)^T P + P^T (A - LI_1)] \epsilon(t) + 2ε^T(t) P^T G \hat{g}(t) - 2ε^T(t) R^T \hat{h}(t)
\]

\[
-2ε^T(t) P^T L_2 \Delta(t) + k_1 \epsilon(t)^T C_1 \hat{e}(t) + k_2 \epsilon(t)^T C_2 \hat{h}(t) + k_3 \epsilon(t)^T C_3 \hat{h}(t)
\]

Since

\[
2ε^T(t) P^T G \hat{g}(t) \leq \frac{1}{κ_2} \epsilon^T(t) P^T GG^T \epsilon(t) + k_3 \epsilon^T(t) \hat{g}(t)
\]

\[
-2ε^T(t) P^T L_2 \hat{h}(t) \leq \frac{1}{κ_2} \epsilon^T(t) P^T L_2^T L_2 P + k_2 \epsilon^T(t) \hat{h}(t)
\]

Using Assumptions 1 and 2, then we can get that

\[
\dot{V}(t) \leq \epsilon^T(t) [(A - LI_1)^T P + P^T (A - LI_1)] + \frac{1}{κ_1} \epsilon^T(t) P^T GG^T \epsilon(t) + \frac{1}{κ_2} \epsilon^T(t) P^T L_2^T L_2 P + k_2 \epsilon^T(t) \hat{h}(t) - 2ε^T(t) P^T L_1 \Delta(t)
\]
Thus, under (14), it can be seen that
\[ \dot{V}(t) \leq -\gamma \| e(t) \|^2 - 2e^T(t)P^T L_2 \Delta(t) \]

Therefore, it can be claimed that
\[ \| e(t) \| \leq \eta_t = \max \{ \| e(0) \|, 2\gamma \| e(t) \| \} \quad (15) \]

As before, \( A_{22} \) is invertible, by Eq.(12) and assumption 1 and 2, we can calculate that
\[ \| e(t) \| \leq \text{max} \{ \| A_{21} \| \| e(t) \|, \| G \| \| e(t) \| \} \leq K(\| A_{21} \| + \| G \| \) \]

It can be seen that
\[ \| e(t) \| \leq \eta_t = \max \{ \| e(0) \|, \| A_{21} \| + \| G \| \} \quad (16) \]

From Eq.(15) and (16), we can see that the error system is stable.

Theorem 1 presents a necessary condition for fault detection. In order to detect \( f \), we select \( e(t) \) as residual signal and propose the following result to determine the threshold.

B. Observer-based fault diagnosis

After the fault is detected based upon the results in section 3.1, the fault diagnosis need to be carried out in order to estimate the size of fault \( f \). When a fault occurs, we construct the following adaptive observer.

\[ \dot{e}(t) = A \dot{\hat{e}}(t) + G \dot{g}(t) + \dot{D}u(t) + L \dot{W} \dot{S}(\hat{x}, u) \]

where
\[ \dot{e}(t) = \text{max} \{ \| e(0) \|, \| A_{21} \| + \| G \| \} \]

The system (4) can be rewritten as
\[ \dot{\hat{e}}(t) = A \hat{e}(t) + G \hat{g}(t) + \dot{D}u(t) + L \dot{W} \dot{S}(\hat{x}, u) \quad (17) \]

The estimated system can be described as
\[ \dot{e}(t) = \dot{\hat{e}}(t) + G \dot{g}(t) + \dot{D}u(t) + L \dot{W} \dot{S}(\hat{x}, u) + \theta(t) \quad (18) \]

Then, an adaptive fault estimation algorithm is presented by the following theorem.

Theorem 2 For the parameters \( \kappa_i > 0 \) (i=1,2), if there exist matrices \( R, P \) with \( P \) being non-singular, and \( \beta > 0 \), such that the following LMI holds:

\[ E^T P + P^T E \geq 0 \quad (19) \]

system (19) with gain \( L = P \dot{R} \) is stable and the fault estimation algorithm is as

\[ \dot{\hat{e}} = -\pi \dot{R}^T P \dot{e} + \dot{e}^T(t) \dot{S}^T (\hat{x}, u) \]

Proof The Lyapunov candidate function can be chosen as
\[ \dot{V}(t) = e(t)^T \dot{P} \dot{e}(t) + e(t + \dot{e}(t))^T \dot{W} \dot{S}(\hat{x}, u) + \dot{e}(t)^T \dot{S}^T (\hat{x}, u) \]

By Eq.(19) and using
\[ \dot{\hat{e}} = -\pi \dot{R}^T P \dot{e} + \dot{e}^T(t) \dot{S}^T (\hat{x}, u) \]

we can obtain
\[ \dot{V}(t) = e(t)^T \dot{P} \dot{e}(t) + e(t + \dot{e}(t))^T \dot{W} \dot{S}(\hat{x}, u) + \dot{e}(t)^T \dot{S}^T (\hat{x}, u) \]

It is noted that
\[ \dot{e}(t)^T \dot{S}^T (\hat{x}, u) = 2e(t)^T \dot{S}^T (\hat{x}, u) \]

Then, we can obtained that
\[ \dot{V}(t) \leq e(t)^T P \dot{e}(t) + e(t + \dot{e}(t))^T \dot{W} \dot{S}(\hat{x}, u) + \dot{e}(t)^T \dot{S}^T (\hat{x}, u) \]

Therefore, it can be shown that
\[ \| e(t) \| \leq \eta_t = \max \{ \| e(0) \|, \| A_{21} \| + \| G \| \} \]

Theorem 2 gives the maximum eigenvalue of \( P \dot{R} \) and \( \dot{R} \).

In the presence of \( \dot{f} \), the following inequality can be obtained
\[ \| e(t) \|^2 \leq \eta_t = \max \{ \| e(0) \|^2, \| \dot{f} \| (\dot{R}^T P \dot{S}) + \dot{e}(t)^T \dot{S}^T (\hat{x}, u) \} \]

From Eq.(12), we have

\[ \| e(t) \| \leq \eta_t = \max \{ \| e(0) \|, \| A_{21} \| + \| G \| \} \]

Eq. (19) with diagnosis observer Eq.(16) based on gain \( L = P \dot{R} \) is stable and the estimation error satisfies
\[ \| e(t) \| \leq \tau_1 + \tau_2^2 \quad (25) \]

If \( \| e(t) \| > \tau_1 + \tau_2^2 \) is satisfied, which indicates that
\[ \| e(t) \| \text{ or } \| e(t) \| \text{ is larger than } \tau_1 \text{ or } \tau_2 \]

Then the error can be made arbitrarily small by choosing the suitable design parameters \( \beta, \beta_1, \beta_2 \text{ and } L \).
IV. CONCLUSION

In this paper, a new FDD method is investigated for singular non-Gaussian stochastic system. It is developed from the technology of PDFs, which is modeled by a square root B-spline expansion. Based on LMI techniques, the complexity FDD problem of singular non-Gaussian stochastic system is transformed into the classical nonlinear FDD problem by introducing the tuning parameter, the corresponding estimation error system is guaranteed to be stable.

REFERENCES