

Networks of Mixed Canonical-Dissipative Systems and Dynamic Hebbian Learning

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Abstract

We study the dynamics of a network consisting of N diffusively coupled, stable-limit-cycle oscillators on which individual frequencies are parametrized by ω_k , $k = 1, \dots, N$. We introduce a learning rule which influences the ω_k by driving the system towards a consensual oscillatory state in which all oscillators share a common frequency ω_c . We are able to analytically calculate ω_c . The network topology strongly affects the relaxation rate but not the ultimate consensual ω_c . We report numerical simulations to show the learning mechanisms at work and confirm our theoretical assertions.

Keywords: mixed canonical-dissipative systems, diffusive coupling, Laplacian matrix, algebraic connectivity.

1. Introduction

In his famous monograph,¹ I.I. Blekhman presents a general formulation of synchronization problems using the following dynamical system:

$$\begin{aligned} \dot{x}_k &= X_k(x_k; \omega_k) + F_k(x_1, \dots, x_N, u), \\ \dot{u}_k &= U(x_1, \dots, x_N, u), \end{aligned} \quad (1)$$

where $X_k = (X_{k,1}, \dots, X_{k,n_k})$, $F_k = (F_{k,1}, \dots, F_{k,n_k})$ and $U = (U_1, \dots, U_v)$ are n_k - and v -dimensional vector fields, respectively. The ω_k are sets of fixed parameters of s_k elements. The F_k and U determine the connections between the local dynamics X_k , $k = 1, \dots, N$. The mathematical formulation in Eqs.(1) covers a broad class of interacting dynamical

systems, ranging from coupled pendulums to complex networks. In this article, we generalize Eps.(1) by proposing the following system:

$$\begin{aligned} \dot{x}_k &= X_k(x_k, \omega_k) + F_k(x_1, \dots, x_N, u), \\ \dot{u}_k &= U(x_1, \dots, x_N, u), \\ \dot{\omega}_k &= \Omega_k(x_1, \dots, x_N, \omega_k), \end{aligned} \quad (2)$$

where $\Omega_k = (\Omega_{k,1}, \dots, \Omega_{k,s_k})$ is a s_k -dimensional vector field determining the dynamics of what was beforehand parameters, and what now becomes state variables ω_k . Indeed, Eqs.(2) allow the parameters ω_k in Eqs.(1) to evolve in time according to Ω_k , and thus to become variables of the system.

Our basic motivation originates from a recent paper² where L. Righetti et al. show how to im-

plement what they call a Dynamic Hebbian Learning (DHL) process by perturbing nonlinear parametric oscillators with an external time-dependent signal. As a paradigmatic illustration, they consider a non-autonomous parametric Hopf oscillator (HO), defined in its phase space by the following system of equations:

$$\begin{aligned} \dot{x} &= +\omega y + (1 - x^2 - y^2)x + \varepsilon \sin(\Omega t), \\ \dot{y} &= -\omega x + (1 - x^2 - y^2)y, \\ \dot{\omega} &= \varepsilon \sin(\theta(t)) \sin(\Omega t), \end{aligned} \quad (3)$$

where ε is a small positive constant, $\sin(\Omega t)$ externally perturbs the basic dynamics of the HO (Ω is a positive constant), and where $\theta(t) := \arctan(\frac{y(t)}{x(t)})$. This is a particular case of Eqs.(2) where the system is reduced to only one local dynamics (the Hopf oscillator), where the connection is given by the perturbing periodic signal, and where the third line of Eqs.(3) determines the dynamics on the frequency parameter ω which is now to be seen as a variable.

The DHL process manifests itself by the fact that the frequency of the underlying HO ω asymptotically approximates Ω , which is the frequency of the external input signal. In other words, the external signal “plastically” deforms the original limit cycle dynamics. We speak about plasticity to reflect the fact that, once this deformation is realized, it subsists even if the external input is removed. This generic behavior can be qualitatively understood by the fact that the external perturbing signal gradually affects the frequency ω on the limit cycle \mathcal{L} (for Eqs.(3), $\mathcal{L} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$), but leaves the shape of \mathcal{L} essentially invariant.

The main aim of the present paper is to substitute in Eqs.(3) the role played by the external signal by the dynamics delivered by other limit cycle oscillators, and to study the resulting mutual DHL process. More generally, we will consider a collection $\{\mathcal{O}_k\}_{k=1}^N$ of independent *mixed canonical-dissipative systems* (MCD),^{3 4 5} which exhibit limit cycles \mathcal{L}_k and different individual $\omega_k, k = 1, 2, \dots, N$ on \mathcal{L}_k . The action of the dissipative mechanism is to stabilize the orbits on \mathcal{L}_k , and the canonic part of the vector field (i.e. its Hamiltonian part) is responsible for the circulation on the limit cycles. In our

class of models the mutual interactions between the \mathcal{O}_k are characterized by:

- a) a network \mathcal{N} of diffusively coupled \mathcal{O}_k - i.e. the row elements of the associated Laplacian coupling matrix of the network add up to zero.
- b) a dynamic Hebbian learning mechanism (DHL). We allow the ω_k to behave as additional variables and implement couplings between these variables and the whole dynamics. Qualitatively speaking, the DHL coupling rule essentially affects the frequency on the limit cycles \mathcal{L}_k while keeping the shape of \mathcal{L}_k approximately unchanged.

The DHL process and the resulting “plasticity” of the dynamics confers a fundamentally different perspective compared to the abundantly studied synchronization networks of limit cycle oscillators. Indeed, interactions of the DHL type offer the possibility to drive the dynamics into a global (identical for all \mathcal{O}_k), stable oscillatory state which, once reached, remains “frozen” even when the interactions are removed. This final oscillatory behavior shared by all \mathcal{O}_k will be called the *consensual oscillatory state*. In this context, a (non-exhaustive) list of issues to be addressed in this paper will read:

- 1) How to calculate the frequency ω_c characterizing the final consensual state ?
- 2) How does the consensual frequency depend on the Laplacian matrix associated to the network ?
- 3) How does the network influence the convergence rate towards the the final consensual state?

In section 2 we propose the construction of an analytically soluble class of coupled oscillators with mutual interactions leading to a DHL rule. A paradigmatic illustration of this class of dynamics is thoroughly studied in section 3, where explicit and fully analytical answers to questions 1) to 3) can be given. Future research perspectives and conclusion will be found in section 5.

2. Construction of a DHL network dynamics

The collection $\{\mathcal{O}_k\}_{k=1}^N$ of oscillators will be chosen to belong to the class of mixed canonical-dissipative

systems which we briefly expose in 2.1.

2.1. Mixed canonical-dissipative systems

A member of our collection $\{\mathcal{O}_k\}_{k=1}^N$ will be defined as:

$$\mathcal{O}_k \begin{cases} \dot{x}_k = +\omega_k \frac{\partial H_k}{\partial y_k} + \underbrace{g_k(H_k) \frac{\partial H_k}{\partial x_k}}_{\text{conservative evolution}}, \\ \dot{y}_k = -\omega_k \frac{\partial H_k}{\partial x_k} + \underbrace{g_k(H_k) \frac{\partial H_k}{\partial y_k}}_{\text{dissipative evolution}}, \end{cases} \quad (4)$$

where $H_k : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ are evaluated at (x_k, y_k) and $g_k : \mathbb{R}^+ \rightarrow \mathbb{R}, k = 1, \dots, N$. The H_k functions are C^2 and positive definite and play the role of Hamiltonians (i.e. energy). Subsequently, we shall assume that $H_k(x_k, y_k) = \mathcal{E}_k$ uniquely defines a set of closed (concentric) curves $\mathcal{L}_k(\mathcal{E}_k)$ in \mathbb{R}^2 surrounding the origin. The g_k functions are C^1 , and $g_k(H_k(x_k, y_k))$ are non-conservative terms which, according to the value of H_k , feed or dissipate energy from the Hamiltonian system. In particular, if $g_k(H_k(x_k, y_k))$ vanishes for $H_k(x_k, y_k) = \mathcal{E}_k$, the dynamics is purely conservative (i.e. only the *canonical* part drives the dynamics) and we have:

$$H_k(x_k, y_k) = \mathcal{E}_k \text{ defines the limit cycle } \mathcal{L}_k(\mathcal{E}_k)$$

with

$$\mathcal{L}_k(\mathcal{E}_k) := \{(x, y) \in \mathbb{R}^2 \mid H_k(x, y) = \mathcal{E}_k\}.$$

The stability of the $\mathcal{L}_k(\mathcal{E}_k)$ will be determined by:

$$\begin{aligned} g_k(H_k) > 0 \text{ in } \mathcal{A}_k & \Rightarrow \mathcal{L}_k(\mathcal{E}_k) \text{ is stable,} \\ g_k(H_k) < 0 \text{ in } \mathbb{R}^2 \setminus \mathcal{A}_k & \\ \\ g_k(H_k) < 0 \text{ in } \mathcal{A}_k & \Rightarrow \mathcal{L}_k(\mathcal{E}_k) \text{ is unstable,} \\ g_k(H_k) > 0 \text{ in } \mathbb{R}^2 \setminus \mathcal{A}_k & \end{aligned} \quad (5)$$

where \mathcal{A}_k stands for the interior of $\mathcal{L}_k(\mathcal{E}_k)$, (i.e. $\mathcal{A}_k := \{(x, y) \in \mathbb{R}^2 \mid H_k(x, y) < \mathcal{E}_k\}$). Therefore, for $g_k(\mathcal{E}_k) = 0$ and when $\mathcal{L}_k(\mathcal{E}_k)$ is stable, the energy-type control $g_k(H_k(x_k, y_k))$ drives all orbits towards the stable limit cycle $\mathcal{L}_k(\mathcal{E}_k)$, which is hence an attractor.

The system defined by Eqs.(4) belongs to the general class of mixed canonical-dissipative dynamics (MCD) (Ref. ^{3, 4} and ⁵). Subsequently, we shall use the short hand notation:

$$\begin{aligned} P_k(x_k, y_k, \omega_k) & := +\omega_k \frac{\partial H_k}{\partial y_k} + g_k(H_k) \frac{\partial H_k}{\partial x_k}, \\ Q_k(x_k, y_k, \omega_k) & := -\omega_k \frac{\partial H_k}{\partial x_k} + g_k(H_k) \frac{\partial H_k}{\partial y_k}, \end{aligned}$$

with evaluation at (x_k, y_k) . Observe that in Eqs.(4), we restrict ourselves to MCD for which ω_k are constant. Having defined the individual dynamics, we can characterize their interactions.

2.2. Network of diffusively coupled oscillators

The interactions between the MCD given by Eqs.(4) will be realized via a simply connected network \mathcal{N} with N vertices and without loops (i.e. its adjacent matrix A is such that, for the j^{th} edge, $A_{j,j} = 0$ and $A_{i,j} \in \{0, 1\}$ for $j \neq i, j = 1, \dots, N$). Let L be the associated Laplacian matrix, namely $L := A - D$, where D is the diagonal matrix with $D_{j,j}$ being the degree of edge j . Accordingly, we now consider the dynamics:

$$\begin{aligned} \dot{x}_k & = \underbrace{P_k(x_k, y_k, \omega_k)}_{\text{MCD}} + \underbrace{C_k x}_{\text{diffusive coupling}} \\ \dot{y}_k & = \underbrace{Q_k(x_k, y_k, \omega_k)}_{\text{MCD}} + \underbrace{C_k y}_{\text{diffusive coupling}} \end{aligned} \quad (6)$$

with $C_k x$ and $C_k y$ reading as:

$$\begin{aligned} C_k x & := \varepsilon_1(x, y) \sum_{j=1}^N L_{k,j} x_j, \\ C_k y & := \varepsilon_2(x, y) \sum_{j=1}^N L_{k,j} y_j, \end{aligned}$$

where $0 \leq \varepsilon_l(x, y) < \varepsilon, l = 1, 2$ not simultaneously vanishing and $x := (x_1, \dots, x_N), y := (y_1, \dots, y_N)$.

Finally, we now introduce the DHL process into the dynamics.

2.3. Dynamic Hebbian learning for mixed canonical-dissipative systems

Directly inspired by Eqs.(3) we now propose our generalized DHL in the context of Eqs.(6). For $k = 1, \dots, N$, the dynamical system is given by:

$$\begin{aligned} \dot{x}_k &= P_k(x_k, y_k, \omega_k) + C_k x, \\ \dot{y}_k &= Q_k(x_k, y_k, \omega_k) + C_k y, \\ \dot{\omega}_k &= \underbrace{K_k [Dy C_k x - Dx C_k y]}_{\text{DHL mechanism}}, \end{aligned} \quad (7)$$

with Dy and Dx reading as:

$$\begin{aligned} Dy &:= \eta_1(x, y) \sum_{j=1}^N \frac{\partial H_j}{\partial y_j}, \\ Dx &:= \eta_2(x, y) \sum_{j=1}^N \frac{\partial H_j}{\partial x_j}. \end{aligned}$$

where $0 \leq K_k \leq \kappa$ is a set of learning coupling strengths, and $0 \leq \eta_l(x, y) \leq \eta$, $l = 1, 2$ are not simultaneously vanishing.

Observe at this point that the dynamics defined by Eqs.(7) exhibits the salient features of the basic model given by Eqs.(3). We have:

- when $C_k x = C_k y = 0$, and for appropriate choices of the $g_k(H_k(x_k, y_k))$ terms, (see Eq.(5)), the local dynamics exhibits a stable limit cycle \mathcal{L}_k ,
- on the limit cycle \mathcal{L}_k , the local dynamics obeys a (conservative) canonical Hamiltonian motion,
- a DHL-type mechanism explicitly affects the frequency ω_k of the orbits on \mathcal{L}_k .

For simplicity and without lost of generality, from now on we shall systematically make $\varepsilon_1(x, y) = \varepsilon_2(x, y) = 1$ and $\eta_1(x, y) = \eta_2(x, y) = 1$ in Eqs.(7).

Proposition 1. *Let $K_k > 0$ for all k in the system defined by Eqs. (7). Then:*

$$J := \sum_{k=1}^N \frac{\omega_k(t)}{K_k} \quad (8)$$

is a constant of the motion.

Proof.

$$\begin{aligned} \sum_{k=1}^N \frac{\dot{\omega}_k}{K_k} &= \sum_{k=1}^N Dy C_k x - \sum_{k=1}^N Dx C_k y \\ &= Dy \sum_{k=1}^N \sum_{j=1}^N L_{kj} x_j - Dx \sum_{k=1}^N \sum_{j=1}^N L_{kj} y_j \\ &= Dy \sum_{j=1}^N x_j \sum_{k=1}^N L_{kj} - Dx \sum_{j=1}^N y_j \sum_{k=1}^N L_{kj} \\ &= 0. \end{aligned}$$

where the last equality identically vanishes due to the diffusive character of the coupling matrix L . \square

Proposition 2. *Assume that we have a collection of identical MCD systems (i.e. $H_k \equiv H$ for all k) admitting, in absence of coupling, the same stable limit cycle $\mathcal{L}_c := \mathcal{L}_k(\mathcal{E}_c)$ for all k (i.e. for a fixed energy level \mathcal{E}_c common to all oscillators, we suppose that $g_k(\mathcal{E}_c) = 0$ for all k). Then, the synchronized orbit given by $\mathcal{S}(t) := (x_s(t), y_s(t), \omega_c, \dots, x_s(t), y_s(t), \omega_c) \in \mathbb{R}^{3N}$, with $\omega_c = \text{constant}$ and with:*

$$g_k(H(x_s(t), y_s(t))) = g_k(\mathcal{E}_c) = 0, \quad (9)$$

is an exact solution of the dynamical system defined by Eqs.(7).

Proof. For the synchronized orbit, we have $x_k(t) = x_s(t)$, $y_k(t) = y_s(t)$ and $\omega_k(t) = \omega_c(t)$ for all k . The diffusive nature of the coupling implies that the terms $C_k x = C_k y = 0$ and therefore $\dot{\omega}_k(t) = 0$. Hence, the $\omega_k(t)$ are all the same constant ω_c . \square

So far, we have introduced a globally non-conservative dynamical system given by Eqs.(7) for which an explicit orbit $\mathcal{S}(t)$ is known. In addition, our dynamics possesses one constant of motion J given by Eq.(8). One may now question whether the orbit $\mathcal{S}(t)$ corresponds to a stable solution of the globally non-conservative dynamics. As usual, linearizing the dynamics around $\mathcal{S}(t)$ provides information regarding its stability - this will be explicitly performed in section 3 for systems with an underlying circular symmetry. To make headway, assume

that $\mathcal{S}(t)$ is indeed a stable solution of the dynamics given by Eqs.(7) and that we have $\lim_{t \rightarrow \infty} \omega_k(t) = \omega_c$ for all k . Hence, ω_c corresponds to the **consensual frequency on the common limit cycle** \mathcal{L}_c . In this case, Proposition 1 and 2 provide explicit answers to questions 1) and 2) raised in the introduction. Indeed, Eq.(8) enables us to write:

$$\text{if } \lim_{t \rightarrow \infty} \omega_k(t) = \omega_c \quad \text{then} \quad \sum_{k=1}^N \frac{\omega_c}{K_k} = \sum_{k=1}^N \frac{\omega_k(0)}{K_k}$$

and therefore, we end with:

$$\omega_c = \frac{\sum_{j=1}^N \frac{\omega_j(0)}{K_j}}{\sum_{j=1}^N \frac{1}{K_j}}. \quad (10)$$

From Eq.(10), we conclude that the consensual frequency ω_c **depends on the distribution of initial conditions $\omega_k(0)$ and on the coupling strength K_k for $k = 1, \dots, N$, but it does not depend on the coupling matrix L and therefore not on the topology of the coupling network.** However, we shall see that L directly affects the convergence rate towards the consensual orbit $\mathcal{S}(t)$.

3. Network of coupled Hopf oscillators

In this section, we focus on a situation where $H_k \equiv H$ for all k and where the underlying Hamiltonian reads as $H(x, y) = H(x^2 + y^2) = H(r^2)$. The circular symmetry implies that the consensual limit cycle \mathcal{L}_c is a circle and the circulation is a uniform rotation with the consensual frequency given by Eq.(10). Due to the cylindrical symmetry, it is advantageous to express the dynamics in polar coordinates:

$$\begin{aligned} \dot{r}_k &= 2(1 - r_k^2)r_k + \sum_{j=1}^N L_{k,j}r_j \cos(\phi_k - \phi_j) \\ \dot{\phi}_k &= -2\omega_k - \frac{1}{r_k} \left(\sum_{j=1}^N L_{k,j}r_j \sin(\phi_k - \phi_j) \right) \\ \dot{\omega}_k &= K_k \left(\sum_{l=1}^N \left(\sum_{j=1}^N L_{k,j}r_l r_j \sin(\phi_l - \phi_j) \right) \right). \end{aligned} \quad (11)$$

Note that in the non-parametric case (i.e. when $\omega_k(t) = \omega_k$), the phase dynamics in Eqs.(11) coincides with the Kuramoto model in the presence of a general coupling network as discussed in ⁶. Here, the exact solution of Eqs. (11) on which perturbations will now be added, simply reads as:

$$\begin{aligned} \mathcal{S}_{pc}(t) &= (r_s(t), \theta_s(t), \omega(t), \dots, r_s(t), \theta_s(t), \omega_s(t)) \\ &= (1, -2\omega_c t, \omega_c, \dots, 1, -2\omega_c t, \omega_c) \in \mathbb{R}^{3N}. \end{aligned} \quad (12)$$

Rearranging the variables in Eqs.(11) by using the permutation $3(k-1) + n \mapsto N(n-1) + k$ ($k = 1, \dots, N$ $n = 1, 2, 3$) and linearizing around $\mathcal{S}_{pc}(t)$ enables us to write:

$$\begin{pmatrix} \dot{\rho} \\ \dot{\delta} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} L - 4\text{Id} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & L & -2\text{Id} \\ \mathbb{O} & -2[K]L & \mathbb{O} \end{pmatrix} \begin{pmatrix} \rho \\ \delta \\ \varepsilon \end{pmatrix} \quad (13)$$

where Id is the identity matrix, $[K]$ is a diagonal matrix with K_1, K_2, \dots, K_N on the diagonal and where $\rho := (\rho_1, \dots, \rho_N)$, $\delta := (\delta_1, \dots, \delta_N)$ and $\varepsilon := (\varepsilon_1, \dots, \varepsilon_N)$ are perturbations. To fulfill the conservation law given by Eq.(8), we further impose the that:

$$\sum_{j=1}^N \frac{\varepsilon_j(0)}{K_j} = 0, \quad (\text{here } K_k \text{ is constant for all } k). \quad (14)$$

To explicitly exhibit the influence of the network, we take a case where $K_k := K$ for all k . Since L is symmetric, there exists an orthogonal matrix V such that $V^T L V$ is a diagonal matrix $[\lambda]$ with its spectrum $\{\lambda_k\}_{k=1}^N$ on the diagonal. The network being connected, there exists a unique j such that λ_j is zero and the rest of the spectrum is strictly negative. Without loss of generality, we assume $\lambda_1 = 0$. Changing the basis of the system by means of a (3×3) -bloc matrix with V^T on its diagonal gives us:

$$\begin{pmatrix} \dot{\tilde{\rho}} \\ \dot{\tilde{\delta}} \\ \dot{\tilde{\varepsilon}} \end{pmatrix} = \begin{pmatrix} [\lambda] - 4\text{Id} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & [\lambda] & -2\text{Id} \\ \mathbb{O} & -2K[\lambda] & \mathbb{O} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{\delta} \\ \tilde{\varepsilon} \end{pmatrix}. \quad (15)$$

The upper left $(N \times N)$ -bloc in Eqs.(15) has N real negative eigenvalues, and the rest of the system is described by the following (2×2) -bloccs:

$$\begin{pmatrix} \tilde{\delta}_k \\ \tilde{\varepsilon}_k \end{pmatrix} = \begin{pmatrix} \lambda_k & -2 \\ -2K\lambda_k & 0 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_k \\ \tilde{\varepsilon}_k \end{pmatrix}. \quad (16)$$

For $k = 1$, we have:

$$\tilde{\delta}_1 = -2\tilde{\varepsilon}_1 \quad \tilde{\varepsilon}_1 = 0.$$

This is a direct consequence of the conservation law Eq.(8) and the restriction imposed by Eq.(14). For $k \neq 1$ the eigenvalues of the System 16 are:

$$\alpha_{\pm} = \frac{1}{2}\lambda_k \pm \frac{1}{2}\sqrt{\lambda_k^2 + 16K\lambda_k}.$$

For a simple connected graph the spectrum $\{\lambda_k\}_{k=1}^N$ is negatively defined⁷ which ensures an (exponential) asymptotic convergence to the consensual state. More precisely, we have a stable focus for $\lambda_k \in]-16K, 0[$ and a stable node for $\lambda_k \in]-\infty, -16K[$. The relaxation time is given by $\tau_{relax} = \mathcal{F}^{-1}$, where \mathcal{F} is the algebraic connectivity (i.e. the *Fiedler number*⁸) of the coupling network. Remember that \mathcal{F} is the largest non-vanishing eigenvalue of the associated Laplacian matrix.

4. Numerical simulations

In Figures 2, 3 and 4 we report numerical simulations performed with five Hopf oscillators (HO) defined by $H(x,y) = x^2 + y^2$ and $g(H) = 1 - H$. Three different topologies of network interactions are considered: “All to All”, “Crystal” and “All to One” (c.f.

Figure 1).

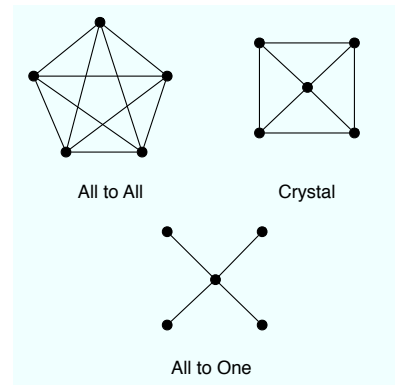


Fig. 1. Three types of network topologies with respective Fiedler number $\mathcal{F}_{(\cdot)}$: “All to All” ($\mathcal{F}_{AtA} = -5$), “Crystal” ($\mathcal{F}_{Cry} = -3$) and “All to One” ($\mathcal{F}_{AtO} = -1$).

The learning mechanism can be observed in Figures 2, 3 and 4, and the final consensual frequency is given by Eq.(8). All three figures have the same time scale, so we can fully appreciate the fact that the convergence rates $\rho_{(\cdot)}$ clearly obey:

$$\mathcal{F}_{AtA} < \mathcal{F}_{Cry} < \mathcal{F}_{AtO} \Rightarrow \rho_{AtA} > \rho_{Cry} > \rho_{AtO}.$$

The smaller the Fiedler number, the faster the convergence, and thus the convergence rate explicitly depends on the topology of the network.

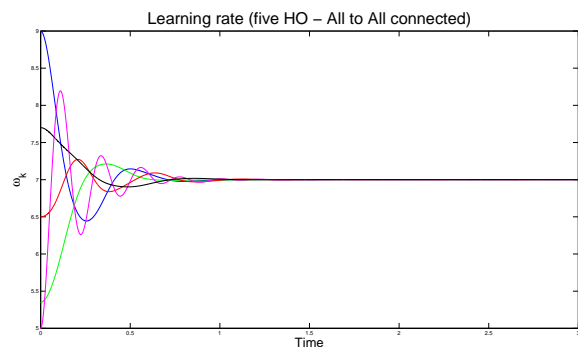


Fig. 2. Time evolution of the frequencies $\omega_k(t)$ for five Hopf oscillators with $K_1 = 1$, $K_2 = \frac{1}{2}$, $K_3 = \frac{5}{4}$, $K_4 = 5$, $K_5 = \frac{1}{3}$ $\omega_1(0) = 9$, $\omega_2(0) = 5.35$, $\omega_3(0) = 6.5$, $\omega_4(0) = 5$, $\omega_5(0) = 7.7$ and with network topology “All to All”. The consensual frequency, given by Eq.(10), is $\omega_c = 7$.

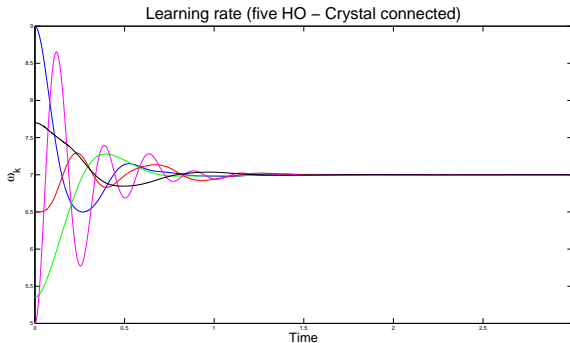


Fig. 3. Time evolution of the frequencies $\omega_k(t)$ for five Hopf oscillators with $K_1 = 1$, $K_2 = \frac{1}{2}$, $K_3 = \frac{5}{4}$, $K_4 = 5$, $K_5 = \frac{1}{3}$ $\omega_1(0) = 9$, $\omega_2(0) = 5.35$, $\omega_3(0) = 6.5$, $\omega_4(0) = 5$, $\omega_5(0) = 7.7$ and with network topology “Crystal”. The consensual frequency, given by Eq.(10), is $\omega_c = 7$.

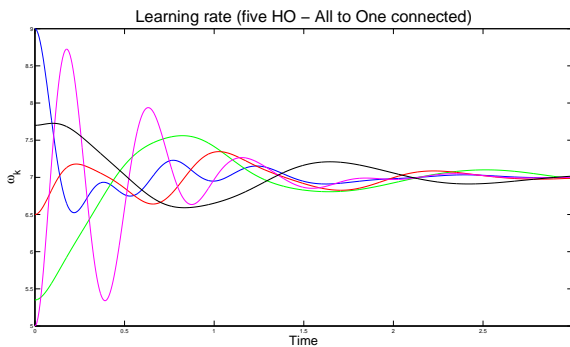


Fig. 4. Time evolution of the frequencies $\omega_k(t)$ for five Hopf oscillators with $K_1 = 1$, $K_2 = \frac{1}{2}$, $K_3 = \frac{5}{4}$, $K_4 = 5$, $K_5 = \frac{1}{3}$ $\omega_1(0) = 9$, $\omega_2(0) = 5.35$, $\omega_3(0) = 6.5$, $\omega_4(0) = 5$, $\omega_5(0) = 7.7$ and with network topology “All to One”. The consensual frequency, given by Eq.(10), is $\omega_c = 7$.

5. Conclusions and Perspectives

Among the numerous possibilities of implementing the DHL learning rule, networks of limit cycle oscillators with adapting frequencies offer a yet unexplored research topics with a vast potential for practical applications. We are here able to explicitly appreciate the interplay between the DHL learning rule

on one hand and the connectivity of the underlying interaction network on the other hand. In particular, the possibility to analytically calculate the consensual frequency (c.f. Eq.(8)) characterizing the circulation of the final common attractor, and the observation that the the topology of the network participate only to the convergence rate are truly remarkable features. At this preliminary stage, we do however not yet offer a complete and mathematically rigorous treatment of the rich underlying dynamics.

Several questions remain open, among them the characterization of the basin of attraction \mathcal{B} of the consensual state, which could be addressed by constructing ad hoc Lyapunov functions. In particular, the dependence of \mathcal{B} on the set of coupling parameters $\{K_k\}_{k=1}^N$, and in the case of coupling networks which can be modeled by multi-edge graphs, remain to be studied.

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